

# THE SHEAR CONSTRUCTION

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**ABSTRACT.** The twist construction is a method to build new interesting examples of geometric structures with torus symmetry from well-known ones. In fact it can be used to construct arbitrary nilmanifolds from tori. In our previous paper, we presented a generalization of the twist, a shear construction of rank one, which allowed us to build certain solvable Lie algebras from  $\mathbb{R}^n$  via several shears. Here, we define the higher rank version of this shear construction using vector bundles with flat connections instead of group actions. We show that this produces any solvable Lie algebra from  $\mathbb{R}^n$  by a succession of shears. We give examples of the shear and discuss in detail how one can obtain certain geometric structures (calibrated  $G_2$ , co-calibrated  $G_2$  and almost semi-Kähler) on two-step solvable Lie algebras by shearing almost Abelian Lie algebras. This discussion yields a classification of calibrated  $G_2$ -structures on Lie algebras of the form  $(\mathfrak{h}_3 \oplus \mathbb{R}^3) \rtimes \mathbb{R}$ .

## 1. INTRODUCTION

The twist construction, introduced in its full generality by the second author in [Swa10], is a geometric model of *T-duality*, a duality relation between different physical theories closely related also to mirror symmetry of Calabi-Yau threefolds [SYZ96]. In particular, the  $S^1$ -version of the twist [Swa07] generalises *T-duality* constructions of Gibbons, Papadopoulos and Stelle [GPS97] in HKT geometry. However, applications of the twist are not restricted to specific geometric structures from physics. In general, one may take an arbitrary tensor field on a manifold  $M$  invariant under the action of a connected  $n$ -dimensional Abelian Lie group  $A$  and twist it to a tensor field on a twist space  $W$  of the same dimension as  $M$ .

The twist considers double fibrations  $M \leftarrow P \rightarrow W$  where both projections are principal  $A$ -bundles and the principal actions commute. An appropriate choice of principal  $A$ -connection gives horizontal spaces  $\mathcal{H}_p$  that are identified with corresponding tangent spaces of  $M$  and  $W$  under the projections. This enables one to transfer any invariant tensor field on  $M$  to a unique tensor field on  $W$ . Moreover, one may recover  $P$  and  $W$  under suitable assumptions from “twist data” on  $M$ , cf. [Swa10]. Hence, one can study properties of the transferred geometric structures solely on  $M$ , without constructing the transferred structure, or even  $W$  or  $P$ , explicitly. This has been successfully applied to produce new interesting examples of various geometric structures from known ones, e.g. examples of SKT, hypercomplex or HKT manifolds with special properties [Swa07] or generalisations of them [FU13, IP13], and includes and generalizes well-known geometric constructions in hyper- and quaternionic Kähler geometry [MS15, Swa16] if one first performs an appropriate “elementary deformation” of the initial structures.

A number of the above examples of the twist are motivated by known results for nilmanifolds. Indeed, we will show that the construction is powerful enough to construct any nilmanifold from the torus by several successive twists. It follows

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that the twist can reproduce all the invariant geometric structures on nilpotent Lie groups or nilmanifolds constructed in the last years, see [BDV09, CF11, Uga07], for example, and may be used to create new ones.

On the other hand, there is much current interest in invariant geometric structures on the larger class of solvable Lie groups and solvmanifolds, see [Fre12, Fre13, CFS11, FV15], for example. But, as we will see, the natural algebraic interpretation of the twist cannot produce these. The aim of this paper is to provide a more general geometric construction that works naturally for solvable groups and solvmanifolds. A rank one version of such a construction was proposed in [FS16], and called the *shear construction*. This was good enough to construct any 1-connected completely solvable Lie groups  $G$  from  $\mathbb{R}^n$  by subsequent shears. Here, we extend the definition of the shear to arbitrary rank and show that this allows one to shear  $\mathbb{R}^n$  to any simply-connected solvable Lie group  $G$  by a sequence of shears. Even for the rank one case this extensions turns out to be slightly more general than that of [FS16].

The main idea is to replace the Abelian group actions in the twist constructions by morphisms of flat vector bundles satisfying a torsion-free condition. Initial data for a shear consists of two flat vector bundles  $E, F$  over  $M$  of equal rank. There should be a vector bundle morphism  $\xi: E \rightarrow TM$ , so the image is locally generated by commuting vector fields that are images of flat sections. Furthermore, there is a two form  $\omega \in \Omega^2(M, F)$  with values in  $F$  satisfying  $d^\nabla \omega = 0$ . One then constructs shears by considering an appropriate submersion  $\pi: P \rightarrow M$  whose vertical subbundle  $\mathcal{V}$  is isomorphic to  $\pi^*F$  and which carries a connection-like one-form  $\theta \in \Omega^1(P, \pi^*F)$  with  $d^\nabla \theta = \pi^*\omega$ . Shears  $S$  of  $M$  are then obtained by lifting  $\xi$  to an injective bundle morphism  $\tilde{\xi}: \pi^*E \rightarrow TP$  and taking  $S$  to be the leaf space of the distribution  $\tilde{\xi}(\pi^*E)$ . As in the twist, there is a pointwise identification of tangent spaces of  $M$  and  $S$  via horizontal spaces in  $P$ , but the tensors that may now be transferred satisfy an invariance condition modified by the connections.

We motivate the particular construction of the shear by first examining the left-invariant situation in detail and understanding the twist construction in this context. In §2.1 we see that the left-invariant twist describes central extensions  $\mathfrak{a}_P \hookrightarrow \mathfrak{p} \twoheadrightarrow \mathfrak{g}$  quotiented by a central ideal  $\tilde{\mathfrak{a}}_G$ . We then demonstrate how one twists  $\mathbb{R}^n$  to any nilpotent Lie algebras by several twists. For the *Lie algebra shear* §2.2, we replace central ideals by arbitrary Abelian ideals and determine the necessary data. This is sufficient to show how to build an arbitrary solvable Lie algebra from  $\mathbb{R}^n$  by a sequence of shears.

In §3.1, we define the general *shear* for arbitrary manifolds equipped with appropriate vector bundles and describe the relevant *shear data*. An important ingredient is a description of the lift procedure for the bundle morphisms  $\xi: E \rightarrow M$  to  $\tilde{\xi}: \pi^*E \rightarrow TP$ , see Theorem 3.5. When  $P$  is a fibre bundle, we note how this may be described via Ehresmann connections. We proceed to determine the conditions for tensor fields on  $M$  to be shear-able to tensor fields on  $W$  and obtain useful formulas for the exterior differentials of sheared forms and Nijenhuis tensors of sheared almost complex structures. Finally, in §3.4 we provide conditions which ensure that the shear is invertible, yielding a dual relationship between  $M$  and  $S$ .

In §4, we apply the shear construction to specific examples. After demonstrating in §4.1 that the fibres of  $\pi$  in the shear construction may vary and that even in the fibre bundle and rank one case,  $\pi$  need not to be a principal bundle in contrast to the findings in [FS16] for our previous definition of a rank one shear, we focus attention on the original situation of left-invariant shears. We explain in §4.2 how our Lie algebra version of the shear considered in §2.2 arises from the general shear of §3.1. We then apply the left-invariant shear to different geometric structures on

*almost Abelian* Lie algebras, i.e. Lie algebras of the form  $\mathbb{R}^n \rtimes \mathbb{R}$ . We concentrate mainly on calibrated and cocalibrated  $G_2$ -structures on seven-dimensional almost Abelian Lie algebras. The two classes of  $G_2$  structures are of interest for various reasons: calibrated  $G_2$ -structures on compact manifolds have interesting curvature properties, e.g. scalar flatness [Bry06] or the Einstein condition [CI07] already imply that they are Ricci-flat; cocalibrated  $G_2$ -structures are structures naturally induced on oriented hypersurfaces in  $Spin(7)$ -manifolds and may, conversely, be used as initial values for the Hitchin flow [Hit01] whose solutions define such manifolds. The almost Abelian cases were classified by the first author in [Fre12, Fre13]. To apply the shear to almost Abelian Lie algebras, we make a natural ansatz for shear data on these Lie algebras and determine when the shear of a calibrated or cocalibrated  $G_2$ -structure is again calibrated or cocalibrated, respectively. In this way, we obtain many explicit examples of such structures on general solvable Lie algebras of step-length two. This leads to a full classification of all calibrated  $G_2$ -structures on Lie algebras of the form  $(\mathfrak{h}_3 \oplus \mathbb{R}^3) \rtimes \mathbb{R}$ . We close the paper with a similar discussion for almost semi-Kähler geometries on solvable Lie algebras.

## 2. LEFT-INVARIANT CONSTRUCTIONS

This section provides detailed motivation for the full definition of the shear construction which will appear in the next section. We describe the twist construction in the setting of left-invariant structures on Lie groups and see how it may be generalized. This involves seeing that the twist in this setting may be considered as first building an extension of the initial Lie algebra  $\mathfrak{g}$  by a *central* Abelian ideal and then constructing the twisted Lie algebra by taking the quotient of this extension by an appropriate central Abelian ideal. To obtain the “shear construction” in this left-invariant setting, we examine what happens when the Abelian ideals are not necessarily central. We then explain how we can apply this kind of shear repeatedly to reduce any solvable Lie algebra to the Abelian Lie algebra  $\mathbb{R}^m$ .

**2.1. A review of the twist construction.** Recall that in general the twist construction [Swa10] considers double fibrations

$$M \xleftarrow{\pi} P \xrightarrow{\pi_W} W.$$

Here each fibration is a principal  $A$ -bundle for some connected  $n$ -dimensional Abelian Lie group  $A$  and the two principal actions on  $P$  are required to commute. It follows that both  $M$  and  $W$  carry actions of the group  $A$ . Furthermore  $P \rightarrow M$  is equipped with a principal  $A$ -connection  $\theta$  which is also invariant under the principal  $A$ -action of  $P \rightarrow W$ . A transversality condition ensures that this connection allows one to relate each  $A$ -invariant differential form  $\alpha$  on  $M$  to a unique differential forms  $\alpha_W$  on  $W$  by requiring that the corresponding pull-backs agree on the horizontal space  $\mathcal{H} := \ker \theta$ .

Under suitable assumptions, one can start with “twist data” on  $M$  and use it to construct first  $P$  and then  $W$ . The twist data is given by:

- (a) an  $A_M \cong A$ -action on  $M$  expressed infinitesimally by a Lie algebra homomorphism  $\xi: \mathfrak{a}_M \rightarrow \mathfrak{X}(M)$ ,
- (b) an  $n$ -dimensional Abelian Lie algebra  $\mathfrak{a}_P$  and a closed integral two-form  $\omega \in \Omega^2(M, \mathfrak{a}_P)$  with values in  $\mathfrak{a}_P$  such that  $\mathcal{L}_\xi \omega = 0$ ,  $\xi^* \omega = 0$  and
- (c) a smooth function  $a: M \rightarrow \mathfrak{a}_P \otimes \mathfrak{a}_M^*$  such that  $\xi \lrcorner \omega = -da$ .

Then  $\pi: P \rightarrow M$  is the principal  $A$ -bundle over  $M$  with connection one-form  $\theta \in \Omega^1(P, \mathfrak{a}_P)$  having curvature  $\pi^* \omega$ . Moreover, if we denote by  $\tilde{\xi}: \mathfrak{a}_M \rightarrow \mathfrak{X}(P)$  the horizontal lift of  $\xi: \mathfrak{a}_M \rightarrow \mathfrak{X}(M)$  and by  $\rho: \mathfrak{a}_P \rightarrow \mathfrak{X}(P)$  the infinitesimal principal action of  $\pi: P \rightarrow M$ , then  $W := P / \langle \tilde{\xi}(\mathfrak{a}_M) \rangle$  for  $\tilde{\xi}: \mathfrak{a}_M \rightarrow \mathfrak{X}(P)$  given by  $\tilde{\xi} = \xi + \rho \circ a$ .

In the left-invariant setting,  $G := M$ ,  $P$  and  $H := W$  are all Lie groups and  $\pi: P \rightarrow G$ ,  $\pi_W: P \rightarrow H$  are Lie group homomorphisms. We may now boil everything down to the associated Lie algebras  $\mathfrak{g}$ ,  $\mathfrak{p}$  and  $\mathfrak{h}$ . The curvature two-form  $\omega$  becomes a closed element of  $\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{a}_P$ , the maps  $\xi: \mathfrak{a}_G \rightarrow \mathfrak{g}$ ,  $\rho: \mathfrak{a}_P \rightarrow \mathfrak{p}$  and  $\tilde{\xi}: \mathfrak{a}_G \rightarrow \mathfrak{p}$  are Lie algebra homomorphisms and  $a$  is constant. Hence,  $\xi \lrcorner \omega = -da = 0$ , which implies that  $\mathcal{L}_\xi \omega = 0$  and  $\xi^* \omega = 0$  hold automatically.

Now let us impose that any element of  $\mathfrak{g}^*$  may be twisted to an element of  $\mathfrak{h}^*$ . This requires  $0 = \mathcal{L}_\xi \alpha = \xi \lrcorner d\alpha = -\alpha([\xi, \cdot])$  for all  $\alpha \in \mathfrak{g}^*$ , which means that  $\xi(\mathfrak{a}_G)$  is central in  $\mathfrak{g}$ . Note that,  $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{a}_P$  as vector spaces with  $\mathfrak{a}_P$  an ideal in  $\mathfrak{p}$ . As  $\mathfrak{g} = \mathcal{H} = \ker \theta$  and as  $d\theta = \pi^* \omega$  for  $\pi: \mathfrak{p} \rightarrow \mathfrak{g}$ , we get  $[\mathfrak{g}, \mathfrak{a}_P] = \{0\}$  and  $[X, Y]_{\mathfrak{p}} = [X, Y]_{\mathfrak{g}} - \omega(X, Y)$  for all  $X, Y \in \mathfrak{g} \subset \mathfrak{p}$ . This means that  $\mathfrak{a}_P$  is central and  $\mathfrak{a}_P \hookrightarrow \mathfrak{p} \twoheadrightarrow \mathfrak{g}$  is a central extension of  $\mathfrak{g}$  by  $\mathfrak{a}_P$ . The extension is determined, up to equivalence, by the Lie algebra cohomology class  $[\omega] \in H^2(\mathfrak{g})$ . Furthermore,  $\tilde{\xi}(\mathfrak{a}_G)$  is a central Abelian ideal in  $\mathfrak{p}$  as  $\tilde{\xi}(\mathfrak{a}_G)$  and  $\mathfrak{a}_P$  are central. Since  $\mathfrak{h} = \mathfrak{p}/\tilde{\xi}(\mathfrak{a}_G)$ ,  $\tilde{\xi}(\mathfrak{a}_G) \hookrightarrow \mathfrak{p} \twoheadrightarrow \mathfrak{h}$  is a central extension as well.

*Remark 2.1.* We will show that we can repeatedly twist any simply-connected  $m$ -dimensional nilpotent Lie group to the Abelian Lie group  $\mathbb{R}^m$ . By duality it will follow that all such nilpotent Lie groups can be constructed by repeatedly twisting from  $\mathbb{R}^m$ .

Let  $N$  be a simply-connected nilpotent Lie group. Write  $\mathfrak{n}$  for the associated  $r$ -step nilpotent Lie algebra and let  $\mathfrak{n}_0 = \mathfrak{n}, \mathfrak{n}_1, \dots, \mathfrak{n}_r = \{0\}$  be the corresponding lower central series of length  $r$ , so  $\mathfrak{n}_1 = \mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}]$  and  $\mathfrak{n}_i = [\mathfrak{n}, \mathfrak{n}_{i-1}]$ . Then  $\mathfrak{n}_{r-1}$  is central and non-zero.

Take  $\mathfrak{a}_P = \mathfrak{a}_G = \mathfrak{n}_{r-1}$  with  $\xi: \mathfrak{a}_G \rightarrow \mathfrak{n}$  the inclusion and  $a: \mathfrak{a}_G \rightarrow \mathfrak{a}_P$  the identity map. Now  $\mathfrak{a} := \mathfrak{a}_P = \mathfrak{a}_G = \mathfrak{n}_{r-1}$  and  $\mathfrak{a} \hookrightarrow \mathfrak{n} \twoheadrightarrow \mathfrak{n}/\mathfrak{a}$  is a central extension of  $\mathfrak{n}/\mathfrak{a}$ . Choosing a linear splitting  $p: \mathfrak{n}/\mathfrak{a} \rightarrow \mathfrak{n}$  gives us a closed two form  $\omega_0 \in \Lambda^2 (\mathfrak{n}/\mathfrak{a})^* \otimes \mathfrak{a}$  defined by  $\omega_0(X, Y) = p[X, Y] - [p(X), p(Y)]$ . Pulling  $\omega_0$  back to  $\mathfrak{n}$ , we obtain a closed two-form  $\omega \in \Lambda^2 \mathfrak{n}^* \otimes \mathfrak{a}$ . This satisfies  $\xi \lrcorner \omega = 0$  and is exact as a smooth form on  $N$ , since  $N$  is diffeomorphic to  $\mathbb{R}^m$ . Hence, we can build a principal  $\mathbb{R}^n$ -bundle  $\pi: P \rightarrow N$ , for  $n = \dim \mathfrak{a}$ , with connection one-form  $\theta \in \Omega^1(P, \mathfrak{a})$  such that  $d\theta = \pi^* \omega$ . The total space  $P$  is also a simply-connected Lie group and  $\theta$  is left-invariant.

As vector spaces, one has  $\mathfrak{p} = \mathfrak{n} \oplus \mathfrak{a}$  and  $\tilde{\xi}(\mathfrak{a}_G) = \Delta(\mathfrak{a})$  is the diagonal in the central subalgebra  $\mathfrak{a} \oplus \mathfrak{a}$  of  $\mathfrak{p}$ . So the twist  $H = P/\Delta(A)$  is a simply-connected Lie group and the associated Lie algebra  $\mathfrak{h}$  has  $\mathfrak{h} \cong \mathfrak{n}/\mathfrak{a} \oplus \mathfrak{a}$  as Lie algebras.

Note that  $\mathfrak{n}/\mathfrak{a} \oplus \mathfrak{a}$  is nilpotent of length  $r - 1$ . Thus iterating this construction, we arrive after  $r - 1$  such twists at the Abelian Lie algebra  $\mathbb{R}^m$ .

**2.2. Shears for Lie algebras.** An obvious generalization of this situation is to consider arbitrary Abelian extensions. To this end, let  $\mathfrak{g}$  be a Lie algebra and let  $\mathfrak{a}_P$  be an Abelian Lie algebra. An extension

$$\mathfrak{a}_P \hookrightarrow \mathfrak{p} \twoheadrightarrow \mathfrak{g}$$

of  $\mathfrak{g}$  by  $\mathfrak{a}_P$  is determined by a two-form  $\omega \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{a}_P$  with values in  $\mathfrak{a}_P$  and a representation  $\eta \in \mathfrak{g}^* \otimes \mathfrak{gl}(\mathfrak{a}_P)$  of  $\mathfrak{g}$  on  $\mathfrak{a}_P$  such that

$$d\omega = -\eta \wedge \omega. \quad (2.1)$$

More precisely, we have  $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{a}_P$  as vector spaces with  $\mathfrak{a}_P$  being an Abelian ideal. The other Lie brackets are given by

$$[X, Y]_{\mathfrak{p}} = [X, Y]_{\mathfrak{g}} - \omega(X, Y) \quad \text{and} \quad [X, Z]_{\mathfrak{p}} = \eta(X)Z,$$

for all  $X, Y \in \mathfrak{g}$  and all  $Z \in \mathfrak{a}_P$ . Note that there is an associated principal  $\mathbb{R}^n$ -bundle  $P \rightarrow G$ , given by the associated simply-connected Lie groups, and a left-invariant

one-form

$$\theta \in \mathfrak{p}^* \otimes \mathfrak{a}_P$$

which is the projection to the  $\mathfrak{a}_P$ -factor in  $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{a}_P$ . Again, we set  $\mathcal{H} := \ker \theta = \mathfrak{g} \subset \mathfrak{p}$  and call  $\mathcal{H}$  the *horizontal* space. Note that

$$d\theta(X, Y) = -\theta([X, Y]_{\mathfrak{p}}) = \omega(X, Y), \quad d\theta(X, Z) = -\eta(X)Z = -\eta(X)(\theta(Z)),$$

for all  $X, Y \in \mathfrak{g} \subset \mathfrak{p}$  and all  $Z \in \mathfrak{a}_P$ . So

$$d\theta = \pi^*\omega - \pi^*\eta \wedge \theta$$

for the projection  $\pi: \mathfrak{p} \rightarrow \mathfrak{g}$ .

Now regard  $\omega$  as a two-form on  $G$  with values in the trivial vector bundle  $F := G \times \mathfrak{a}_P$  and  $\eta$  as a connection  $\nabla$  on  $F$  in the sense that  $\nabla_X f = X(f) + \eta(X)(f)$  for all vector fields  $X$  on  $G$  and all sections  $f$  of  $F$ , regarded as smooth functions  $f: G \rightarrow \mathfrak{a}_P$ . The condition that  $\eta$  is a representation is equivalent to  $\nabla$  being flat. In the case of a central extension,  $\nabla$  is just the natural flat connection on the trivial bundle  $F = G \times \mathfrak{a}_P$ .

The flatness of the connection  $\nabla$  implies that the associated exterior covariant derivative  $d^\nabla: \Omega^k(G, F) \rightarrow \Omega^{k+1}(G, F)$  squares to 0. As this derivative satisfies  $d^\nabla \alpha = \eta \wedge \alpha + d\alpha$  for all  $\alpha \in \Omega^k(G, F) = \Omega^k(G, \mathfrak{a}_P)$ , we have

$$d^\nabla \omega = 0$$

from (2.1).

We may consider  $\theta$  as a left-invariant one-form on  $P$  with values in the trivial bundle  $P \times \mathfrak{a}_P$ . This trivial bundle is the pull-back of  $F$ , so we can pull the connection  $\nabla$  back to  $P \times \mathfrak{a}_P$ . We denote the resulting connection by  $\nabla$  too. Then we have

$$d^\nabla \theta = \pi^*\omega.$$

By analogy with the twist, we wish to construct a new Lie algebra homomorphism

$$\overset{\circ}{\xi}: \mathfrak{a}_G \rightarrow \mathfrak{p},$$

from an Abelian Lie algebra  $\mathfrak{a}_G$  of the same dimension as  $\mathfrak{a}_P$ , with the property that  $\overset{\circ}{\xi}(\mathfrak{a}_G)$  is an  $n$ -dimensional Abelian ideal in  $\mathfrak{p}$ . We will then define

$$\mathfrak{h} := \mathfrak{p} / \overset{\circ}{\xi}(\mathfrak{a}_G)$$

to be the *shear* of  $\mathfrak{g}$ . The associated simply-connected Lie groups will then give us a principal  $\mathbb{R}^n$ -bundle  $P \rightarrow H$  and so a double fibration

$$G \longleftarrow P \longrightarrow H.$$

As in the twist, the construction of  $\overset{\circ}{\xi}$  should arise from a Lie algebra homomorphism  $\xi: \mathfrak{a}_G \rightarrow \mathfrak{g}$  with  $\xi = \pi \circ \overset{\circ}{\xi}$ , which we require to be injective as in the twist case this corresponds to the action being effective. We can write

$$\overset{\circ}{\xi} = \tilde{\xi} + \rho \circ a,$$

where  $\tilde{\xi}: \mathfrak{a}_G \rightarrow \mathcal{H} = \mathfrak{g} \subset \mathfrak{p}$  is the horizontal lift and  $\rho: \mathfrak{a}_P \rightarrow \mathfrak{p}$  is the inclusion. We will require the map  $a: \mathfrak{a}_G \rightarrow \mathfrak{a}_P$  to be an isomorphism of Lie algebras. In the vector space splitting  $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{a}_P$ , the prescription for  $\overset{\circ}{\xi}$  just reads  $\overset{\circ}{\xi}Z = (\xi Z, aZ)$  for  $Z \in \mathfrak{a}_G$ .

We need some notations for stating our results on the existence of  $\overset{\circ}{\xi}$ . The map  $\xi: \mathfrak{a}_G \rightarrow \mathfrak{g}$  may be considered as a bundle morphism  $\xi: E \rightarrow TG$ , where  $E := G \times \mathfrak{a}_G$  is the trivial bundle. This carries a connection given by

$$\gamma := a^{-1}(\xi \lrcorner \omega) + a^{-1}\eta a \in \mathfrak{g}^* \otimes \mathfrak{gl}(\mathfrak{a}_G). \quad (2.2)$$

We now have induced connections on all bundles of the form  $E^{\otimes r} \otimes F^{\otimes s}$  for  $r, s \in \mathbb{Z}$ . This gives associated covariant exterior derivatives for  $k$ -forms on  $G$  with values in

these bundles. To simplify the notation, we denote all these connections by  $\nabla$  and the associated covariant exterior derivatives by  $d^\nabla$ . For any  $k$ -form  $\beta$  on  $G$  with values in such a bundle, we set

$$\mathcal{L}_\xi^\nabla \beta := d^\nabla(\xi \lrcorner \beta) + \xi \lrcorner d^\nabla \beta.$$

**Lemma 2.2.** *Let  $\xi: \mathfrak{a}_G \rightarrow \mathfrak{g}$  be an injective Lie algebra homomorphism. Then the map  $\xi: \mathfrak{a}_G \rightarrow \mathfrak{p}$  defined as above is a Lie algebra homomorphism with image an Abelian ideal if and only if*

- (i)  $[\xi e_1, \xi e_2] = \xi(\nabla_{\xi e_1} e_2 - \nabla_{\xi e_2} e_1)$  for all  $e_1, e_2 \in \Gamma(E)$ ,
- (ii)  $\xi^* \omega = 0$ ,
- (iii)  $\mathcal{L}_\xi^\nabla \alpha = 0$  for all  $\alpha \in \mathfrak{g}^*$ .

*If these conditions are true, then all connections  $\nabla$  are flat, so  $(d^\nabla)^2 = 0$ , and*

$$d^\nabla a = -\xi \lrcorner \omega, \quad \mathcal{L}_\xi^\nabla \omega = 0, \quad \mathcal{L}_\xi^\nabla \theta = 0.$$

*Proof.* The condition that  $\xi$  is a Lie algebra homomorphism is equivalent to  $\xi(\mathfrak{a}_G)$  being Abelian. For  $Z, W \in \mathfrak{a}_G$ , this says

$$\begin{aligned} (0, 0) &= [\xi Z, \xi W]_{\mathfrak{p}} = [(\xi Z, aZ), (\xi W, aW)]_{\mathfrak{p}} \\ &= (0, -\omega(\xi Z, \xi W) + \eta(\xi Z)(aW) - \eta(\xi W)(aZ)), \end{aligned}$$

which implies that  $\xi^* \omega(Z, W) = \omega(\xi Z, \xi W) = \eta(\xi Z)(aW) - \eta(\xi W)(aZ)$  is equivalent to  $\xi(\mathfrak{a}_G)$  being Abelian.

To investigate the condition that  $\xi(\mathfrak{a}_G)$  is an ideal, we compute

$$\begin{aligned} [(X, 0), \xi Z]_{\mathfrak{p}} &= [(X, 0), (\xi Z, aZ)]_{\mathfrak{p}} = ([X, \xi Z]_{\mathfrak{g}}, -\omega(X, \xi Z) + \eta(X)(aZ)), \\ [(0, Y), \xi Z]_{\mathfrak{p}} &= [(0, Y), (\xi Z, aZ)]_{\mathfrak{p}} = (0, -\eta(\xi Z)Y) \end{aligned}$$

for  $X \in \mathfrak{g}$ ,  $Y \in \mathfrak{a}_P$  and  $Z \in \mathfrak{a}_G$ . Thus  $\xi(\mathfrak{a}_G)$  is an ideal in  $\mathfrak{p}$  if and only if  $[\xi Z, X]_{\mathfrak{g}} = -\xi a^{-1}(\omega(\xi Z, X) + \eta(X)(aZ)) = -\xi(\gamma(X)Z)$  and  $0 = \xi(a^{-1}\eta(\xi Z)(Y))$ . As  $\xi$  is injective, the latter equation is equivalent to  $\eta(\xi Z)(Y) = 0$  for all  $Y \in \mathfrak{a}_P$  and  $Z \in \mathfrak{a}_G$ . Moreover, for  $\alpha \in \mathfrak{g}^*$ , we have

$$\begin{aligned} (\mathcal{L}_\xi^\nabla \alpha)(Z, X) &= (\xi(Z) \lrcorner d\alpha)(X) + d^\nabla(\xi \lrcorner \alpha)(Z, X) = -\alpha([\xi Z, X]_{\mathfrak{g}}) - \alpha(\xi \nabla_X Z) \\ &= -\alpha([\xi Z, X]_{\mathfrak{g}}) + \xi(\gamma(X)Z) \end{aligned}$$

for all  $X \in \mathfrak{g}$  and all  $Z \in \mathfrak{a}_G$ . Hence,  $\xi(\mathfrak{a}_G)$  is an Abelian ideal if and only if (ii) and (iii) from the statement hold and  $\eta(\xi Z)Y = 0$  for all  $Y \in \mathfrak{a}_P$  and  $Z \in \mathfrak{a}_G$  holds.

So assume now that (ii) and (iii) are true. If then also  $\eta(\xi Z)Y = 0$  holds for any  $Y \in \mathfrak{a}_P$ ,  $Z \in \mathfrak{a}_G$ , condition (ii) implies  $\gamma(\xi Z)(Y) = 0$  and so  $[\xi(Z), \xi(Y)] = 0 = \xi(\nabla_{\xi(Z)} Y - \nabla_{\xi(Y)} Z)$  for all  $Z, Y \in \mathfrak{a}_G$ . Hence, condition (i) holds. Conversely, if additionally (i) is valid, we obtain

$$-\xi(\gamma(\xi Y)X) = [\xi(X), \xi(Y)] = \xi(\nabla_{\xi X} Y - \nabla_{\xi Y} X) = \xi(\gamma(\xi X)Y - \gamma(\xi Y)X)$$

and so  $\gamma(\xi X)Y = 0$  for all  $X, Y \in \mathfrak{a}_G$ . But then condition (ii) implies  $\eta(\xi Y)Z = 0$  for all  $Y \in \mathfrak{a}_P$ ,  $Z \in \mathfrak{a}_G$ .

Now if (i)–(iii) are true, we have  $[\xi(\cdot), X] = -\xi \circ \gamma(X)$  for all  $X \in \mathfrak{g}$  and the Jacobi identity gives us  $[\gamma(X), \gamma(Y)] = \gamma([X, Y])$  for all  $X, Y \in \mathfrak{g}$ . Thus  $\nabla$  is flat on  $E$  and so the induced connections  $\nabla$  on bundles of the form  $E^{\otimes r} \otimes F^{\otimes s}$  are flat as well. In particular,  $(d^\nabla)^2 = 0$ . Now  $(d^\nabla a)(Y, X) = \nabla_X(a(Y)) - a(\nabla_X Y) = \eta(X)aY - a\gamma(X)Y = -\omega(\xi(Y), X)$  for all  $X \in \mathfrak{g}$  and all  $Y \in \mathfrak{a}_G$ , i.e.  $d^\nabla a = -\xi \lrcorner \omega$ . But then  $\mathcal{L}_\xi^\nabla \omega = \xi \lrcorner d^\nabla \omega + d^\nabla(\xi \lrcorner \omega) = -(d^\nabla)^2 a = 0$ . Finally,  $\mathcal{L}_\xi^\nabla \theta = \xi \lrcorner \pi^* \omega + d^\nabla \pi^* a = \pi^*(\xi \lrcorner \omega + d^\nabla a) = 0$ .  $\square$

*Remark 2.3.* The proof of Lemma 2.2 shows that for non-injective  $\xi: \mathfrak{a}_G \rightarrow \mathfrak{g}$ , conditions (i)–(iii) in Lemma 2.2 still imply that  $\check{\xi}(\mathfrak{a}_G)$  is an ideal in  $\mathfrak{p}$ , but not necessarily Abelian anymore.

**Definition 2.4.** If in the situation of Lemma 2.2 the conditions (i)–(iii) are true, then we build the Lie algebra  $\mathfrak{h} := \mathfrak{g}/\check{\xi}(\mathfrak{a}_G)$  and call it the *shear* of  $\mathfrak{g}$ .

**Proposition 2.5.** *Any two solvable Lie algebras of the same dimension are related via a sequence of shear constructions.*

*Proof.* It is enough to show how to relate any solvable algebra to the Abelian algebra of the same dimension.

Let  $\mathfrak{s}$  be an  $r$ -step solvable Lie algebra  $r$ -step of dimension  $n$ . We will obtain the Abelian algebra  $\mathbb{R}^n$  by a series of  $r - 1$  shears. Let  $\mathfrak{s}^{(0)} = \mathfrak{s}, \dots, \mathfrak{s}^{(r)} = [\mathfrak{s}^{(r-1)}, \mathfrak{s}^{(r-1)}] = \{0\}$  be the derived series of  $\mathfrak{s}$ . Then  $\mathfrak{a} := \mathfrak{s}^{(r-1)}$  is an Abelian ideal in  $\mathfrak{s}$ . We take  $\mathfrak{a}_G = \mathfrak{a}_P = \mathfrak{a}$ , let  $\xi$  be the inclusion, take  $a: \mathfrak{a}_G \rightarrow \mathfrak{a}_P$  to be the identity map and use the canonical flat connection  $\eta = 0$  on  $F := G \times \mathfrak{a}_P$ .

Choose a vector space splitting  $p: \mathfrak{g}/\mathfrak{a} \rightarrow \mathfrak{g}$  of  $\mathfrak{a} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{a}$ . This induces a projection  $\pi_{\mathfrak{a}}: \mathfrak{g} \rightarrow \mathfrak{a}$  with kernel  $p(\mathfrak{g}/\mathfrak{a})$ . Let  $\omega \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{a}$  be the negative of projection of the Lie bracket to  $\mathfrak{a}$ , so  $\omega(X, Y) = -\pi_{\mathfrak{a}}[X, Y]$ . Then  $d^\nabla \omega = d\omega = 0$  by the Jacobi identity. Moreover,  $\xi^* \omega = 0$  as  $\mathfrak{a}$  is an Abelian ideal in  $\mathfrak{g}$ . Now  $\gamma = a^{-1}(\xi \lrcorner \omega) + a^{-1} \eta a = \xi \lrcorner \omega$  gives  $\nabla_{\xi X} Y = \gamma(\xi X)(Y) = \omega(\xi Y, \xi X) = 0$  for all  $X, Y \in \mathfrak{a}_G$ , so condition (i) in Lemma 2.2 is satisfied. Finally, for  $X \in \mathfrak{a}_G$  and  $Y \in \mathfrak{g}$ , we have  $\gamma(Y)(X) = \omega(\xi X, Y) = -[\xi X, Y]$  since  $\mathfrak{a} = \xi(\mathfrak{a}_G)$  is an ideal. By the proof of Lemma 2.2, this is equivalent to condition (iii) from Lemma 2.2.

We may then use this data to shear  $\mathfrak{g}$  to the solvable Lie algebra  $\mathfrak{h} = \mathfrak{p}/\check{\xi}(\mathfrak{a})$ . Now  $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{a}$  as vector spaces and  $\check{\xi}Z = (Z, Z) \in \mathfrak{p} = \mathfrak{g} \oplus \mathfrak{a}$  for  $Z \in \mathfrak{a}$ . It follows that  $\mathfrak{h}$  is the Lie algebra direct sum  $\mathfrak{h} = (\mathfrak{g}/\mathfrak{a}) \oplus \mathfrak{a}$ , since for  $A, B \in \mathfrak{g}$ ,  $[A, B]_{\mathfrak{p}} = ([A, B], \pi_{\mathfrak{a}}([A, B])) \equiv ((1 - \pi_{\mathfrak{a}})[A, B], 0) \bmod \check{\xi}(\mathfrak{a})$ . In particular the shear  $\mathfrak{h}$  is solvable of step length  $(r - 1)$ . Iterating the construction, after  $r - 1$  shears we arrive at the Abelian Lie algebra  $\mathbb{R}^m$ , as claimed.

Conversely, suppose we are given  $\mathfrak{h} = (\mathfrak{g}/\mathfrak{a}) \oplus \mathfrak{a}$ . Then  $k: \mathfrak{h} = (\mathfrak{g}/\mathfrak{a}) \oplus \mathfrak{a} \rightarrow \mathfrak{g}$ ,  $k((X, Y)) := p(X) + Y$  is a vector space isomorphism. A shear that recovers  $\mathfrak{g}$  is now given by the two-form  $\tilde{\omega} := k^* \omega \in \Lambda^2 \mathfrak{h}^* \otimes \mathfrak{a}$ , the one-form  $\tilde{\eta} := k^* \gamma \in \mathfrak{h}^* \otimes \mathfrak{gl}(\mathfrak{a})$ ,  $\tilde{\xi} := -\text{inc}$  and  $\tilde{a} := \text{id}_{\mathfrak{a}}$ , cf. also Theorem 3.14  $\square$

### 3. THE SHEAR

**3.1. Lifting certain vector bundle morphisms.** Now we define the shear construction in full generality. Motivated by the last section, we start with a vector bundle  $\pi_E: E \rightarrow M$  endowed with a flat connection  $\nabla = \nabla^E$  and a vector bundle morphism  $\xi: E \rightarrow TM$  satisfying condition (i) above, that is

$$\xi(\nabla_{\xi e_1} e_2 - \nabla_{\xi e_2} e_1) = [\xi e_1, \xi e_2]. \quad (3.1)$$

We will then say that  $(\xi, \nabla)$  is *torsion free*. Moreover, we assume that we have a second vector bundle  $\pi_F: F \rightarrow M$  of the same rank with flat connection  $\nabla = \nabla^F$  and a two-form  $\omega \in \Omega^2(M, F)$  with values in  $F$  such that  $d^\nabla \omega = 0$ . We do not require condition (ii) here as it will naturally follow from our set-up below. Condition (iii) will arise as the appropriate invariance condition when we consider transferring differential forms in §3.2.

Let us assume that  $M$  is the leaf space of a foliation on some manifold  $P$  with leaves of dimension  $\text{rk}(E) = \text{rk}(F)$ . Write  $\pi: P \rightarrow M$  for the projection, which is a surjective submersion. We wish to identify the pull-back of  $F$  to  $P$  with the tangent spaces to the leaves of the foliation.

**Definition 3.1.** Suppose there are a vector bundle morphism  $\rho: \pi^*F \rightarrow TP$  and a one-form  $\theta \in \Omega^1(P, \pi^*F)$ , so a bundle morphism  $\theta: TP \rightarrow \pi^*F$ , such that

- (1)  $\theta \circ \rho = \text{id}_{\pi^*F}$ ,
- (2)  $d\pi \circ \rho = 0$  and
- (3)  $d^\nabla \theta = \pi^*\omega$ .

Then we call  $(P, \theta, \rho)$  a *shear total space* for  $\omega$ . We will call  $\dim P - \dim M$  the *rank* of  $P$ .

It follows that the dimension of each leaf of the foliation on  $P$  is equal to the rank of  $F$ . We define the natural subbundles

$$\mathcal{H} := \ker \theta, \quad \mathcal{V} := \ker d\pi$$

of  $TP$ . We note that our assumptions give  $\mathcal{V} = \rho(\pi^*F)$ , so  $TP = \mathcal{H} \oplus \mathcal{V}$ . We call  $\mathcal{H}$  the *horizontal* and  $\mathcal{V}$  the *vertical* subbundles.

*Remark 3.2.* For a shear total space, conditions (1) and (2) identify the vertical subbundle  $\mathcal{V}$  with the flat vector bundle  $\pi^*F$ . In particular, the element  $\hat{\theta} = \rho \circ \theta$  in  $\Omega^1(P, \mathcal{V}) = \text{Hom}(TP, \mathcal{V}) \subset \text{End}(TP)$  is a projection onto the vertical subbundle  $\mathcal{V}$ . Thus, if  $P$  is actually a fibre bundle, then  $\hat{\theta}$  is (the connection form of) an Ehresmann connection on  $P$ . Recall that the *curvature*  $R \in \Omega^2(P, TP)$  of the Ehresmann connection  $\hat{\theta}$  is  $R(X, Y) = \hat{\theta}[X_H, Y_H]$  for all  $X, Y \in \mathfrak{X}(P)$ , where  $Z_H$  is the horizontal part of  $Z \in \mathfrak{X}(P)$ . Hence, condition (3) implies  $R = -\rho \circ \pi^*\omega$  since both forms are horizontal and  $d^\nabla \theta(X, Y) = \nabla_X(\theta Y) - \nabla_Y(\theta X) - \theta[X, Y] = -\theta[X, Y]$  for horizontal  $X$  and  $Y$ .

Suppose now that we are given an arbitrary Ehresmann connection  $\hat{\theta}$  on a fibre bundle  $\pi: P \rightarrow M$  as in Remark 3.2. Then the natural question arises when  $\hat{\theta}$  gives rise to a shear total space  $(P, \theta, \text{inc})$ . This question is answered in the following

**Proposition 3.3.** *Let  $\pi: P \rightarrow M$  be a fibre bundle such that the vertical subbundle  $\mathcal{V}$  is the pull-back of a flat vector bundle  $(F, \nabla)$  over  $M$ . Write  $\text{inc}: \pi^*F = \mathcal{V} \rightarrow TP$  for the inclusion map. Then an Ehresmann connection  $\hat{\theta} \in \text{End}(TP)$  gives rise to a shear total space  $(P, \theta, \text{inc})$ ,  $\hat{\theta} = \text{inc} \circ \theta$ , for some  $\omega \in \Omega^2(M, F)$ , if and only if*

$$[X_1, X_2] = \nabla_{X_1} X_2 - \nabla_{X_2} X_1 \tag{3.2}$$

for all vertical  $X_1, X_2 \in \mathfrak{X}(P)$  and all local parallel vertical vector fields preserve the horizontal subbundle  $\mathcal{H}$ . If this is the case, then the curvature  $R$  of  $\hat{\theta}$  satisfies  $d^\nabla \theta = \pi^*\omega = -R$ .

*Proof.* Note that there is an  $\omega \in \Omega^2(M, F)$  with  $d^\nabla \theta = \pi^*\omega$  if and only if for all local parallel frames  $(f^1, \dots, f^k)$  of  $F$  the forms  $f^1 \circ d^\nabla \theta, \dots, f^k \circ d^\nabla \theta$  are basic. As  $d(f^i \circ d^\nabla \theta) = f^i \circ d^\nabla d^\nabla \theta = 0$  for all  $i = 1, \dots, k$ , this is, in turn, equivalent to  $d^\nabla \theta$  being horizontal.

To check the horizontality of  $d^\nabla \theta$ , first let  $X_1, X_2$  be two vertical vector fields on  $P$ . Then

$$\begin{aligned} d^\nabla \theta(X_1, X_2) &= \nabla_{X_1}(\theta X_2) - \nabla_{X_2}(\theta X_1) - \theta([X_1, X_2]) \\ &= \nabla_{X_1} X_2 - \nabla_{X_2} X_1 - [X_1, X_2] \end{aligned}$$

as  $[X_1, X_2]$  is vertical. So  $d^\nabla \theta(X_1, X_2) = 0$  if and only if equation (3.2) holds. Next, let  $X, Y$  be vector fields on  $P$  with  $X$  vertical and  $Y$  horizontal. As  $\mathcal{V} = \pi^*F$  has a local basis of parallel sections, we may assume that  $X$  is parallel. Hence,  $d^\nabla \theta(X, Y) = -\theta([X, Y])$ , which is zero if and only if  $\mathcal{L}_X Y = [X, Y]$  is horizontal, i.e. if and only if  $X$  preserves the horizontal subbundle.  $\square$



*Remark 3.4.* By equation (3.2), the fibres of  $\pi: P \rightarrow M$  are endowed with a torsion-free flat connection, so they are affine manifolds. Moreover,  $\mathcal{V}$  is the pull-back of a vector bundle over  $M$ , so the fibres are parallelisable. As the connection is a pull-back, there is a parallel, and so commuting, basis of vector fields on each fibre. This is in accordance with the twist case where the fibres were connected Abelian Lie groups.

The proof of Proposition 3.3 shows that for an arbitrary shear total space  $(P, \theta, \rho)$  we have  $[\rho f_1, \rho f_2] = \rho(\nabla_{\rho f_1} f_2 - \nabla_{\rho f_2} f_1)$  for all  $f_1, f_2 \in \Gamma(\pi^* F)$ , which is the torsion-free condition (3.1). Finally, note that the condition that (local) parallel vertical vector fields preserve the horizontal subbundle in Proposition 3.3 corresponds in the twist case to the principal action preserving the horizontal subbundle.

Now we want to find conditions under which there exists a vector bundle morphism  $\overset{\circ}{\xi}: \pi^* E \rightarrow TP$  covering  $\xi$ , i.e.

$$\begin{array}{ccc} \pi^* E & \xrightarrow{\overset{\circ}{\xi}} & TP \\ \downarrow & & \downarrow d\pi \\ E & \xrightarrow{\xi} & TM. \end{array} \quad (3.3)$$

commutes, such that  $\overset{\circ}{\xi}$  preserves  $\theta$ :

$$\mathcal{L}_{\overset{\circ}{\xi}}^\nabla \theta := d^\nabla(\overset{\circ}{\xi} \lrcorner \theta) + \overset{\circ}{\xi} \lrcorner d^\nabla \theta = 0 \quad (3.4)$$

and  $(\overset{\circ}{\xi}, \nabla)$  is torsion-free:

$$\overset{\circ}{\xi}(\nabla_{\overset{\circ}{\xi} \tilde{e}_1} \tilde{e}_2 - \nabla_{\overset{\circ}{\xi} \tilde{e}_2} \tilde{e}_1) = [\overset{\circ}{\xi} \tilde{e}_1, \overset{\circ}{\xi} \tilde{e}_2] \quad (3.5)$$

for all  $\tilde{e}_1, \tilde{e}_2 \in \Gamma(\pi^* E)$ .

Let us motivate our interest in such a bundle map  $\overset{\circ}{\xi}$ . Firstly, the two conditions were true in the left-invariant case discussed in the previous section and they hold for the map  $\rho$  of a shear total space, since  $\mathcal{L}_\rho^\nabla \theta = \rho \lrcorner d^\nabla \theta + d^\nabla \text{id}_{\pi^* F} = \rho \lrcorner \pi^* \omega = 0$ . Furthermore, the torsion-free condition implies that  $\overset{\circ}{\xi}(\pi^* E)$  is involutive; we will see that it is equivalent to involutivity under our assumptions. When  $\overset{\circ}{\xi}$  has constant rank, this is, in turn, equivalent to the integrability of  $\overset{\circ}{\xi}(\pi^* E)$ . The shear should then be the leaf space of the corresponding foliation on  $P$ .

For the first condition, note that  $\mathcal{L}_{\overset{\circ}{\xi}}^\nabla \theta = 0$  may be written as

$$\nabla_{\overset{\circ}{\xi} \tilde{e}}(\theta X) = \theta(\overset{\circ}{\xi}(\nabla_X \tilde{e}) + [\overset{\circ}{\xi} \tilde{e}, X]) \quad (3.6)$$

for all vector fields  $X \in \mathfrak{X}(P)$  and sections  $\tilde{e} \in \Gamma(\pi^* E)$ . As  $E$  and  $F$  are flat, they have local bases of parallel sections. Take  $\tilde{e} = \pi^* e$  for some local parallel section  $e$  of  $E$ . If  $X$  is horizontal, this gives  $\theta([\overset{\circ}{\xi}(\pi^* e), X]) = 0$ , so  $\overset{\circ}{\xi}(\pi^* e)$  preserves the horizontal space, just as the lifted action does in the twist construction. If  $X$  is a local vertical vector field with  $\theta X$  is parallel, we obtain that  $[\overset{\circ}{\xi}(\pi^* e), X]$  is horizontal. However,  $\overset{\circ}{\xi}(\pi^* e)$  is  $\pi$ -related to  $\xi e$  and  $X$  is  $\pi$ -related to 0, so the commutator has to be vertical too. This shows

$$[\overset{\circ}{\xi}(\pi^* e), X] = 0, \quad (3.7)$$

which in the twist construction is the requirement that the lifted and principal actions commute.

After this motivation, we are interested in expressing the requirements for a lift  $\overset{\circ}{\xi}$  as above in equivalent conditions for data on  $M$ .

**Theorem 3.5.** *Under the above assumptions (3.1) and Definition 3.1(1)–(3), there exists a vector bundle morphism  $\overset{\circ}{\xi}: \pi^* E \rightarrow TP$  covering  $\xi: E \rightarrow TM$ , preserving  $\theta$  and with  $(\overset{\circ}{\xi}, \nabla)$  torsion-free if and only if  $\mathcal{L}_\xi^\nabla \omega = 0$ ,  $\xi \lrcorner \omega$  is  $d^\nabla$ -exact and  $\xi^* \omega = 0$ .*

*Proof.* Using  $\mathcal{H}$ , we can lift  $\xi: E \rightarrow TP$  uniquely to a bundle morphism  $\tilde{\xi}: \pi^*E \rightarrow TP$  covering  $\xi$  with  $\tilde{\xi}(\pi^*E) \subset \mathcal{H}$ . When  $\tilde{\xi}$  exists we get  $d\pi(\tilde{\xi} - \tilde{\xi}) = 0$ , so  $(\tilde{\xi} - \tilde{\xi})(\pi^*E) \subset \mathcal{V}$ . As  $\rho: \pi^*F \rightarrow TP$  is injective and  $\rho(\pi^*F) = \mathcal{V}$ , there is a uniquely defined bundle map  $\tilde{a}: \pi^*E \rightarrow \pi^*F$  with

$$\tilde{\xi} = \tilde{\xi} + \rho \circ \tilde{a}. \quad (3.8)$$

Thus,

$$\mathcal{L}_{\tilde{\xi}}^\nabla \theta = d^\nabla(\tilde{\xi} \lrcorner \theta) + \tilde{\xi} \lrcorner d^\nabla \theta = d^\nabla \tilde{a} + \tilde{\xi} \lrcorner \pi^* \omega = d^\nabla \tilde{a} + \pi^*(\xi \lrcorner \omega),$$

and so  $\mathcal{L}_{\tilde{\xi}}^\nabla \theta = 0$  if and only if  $\pi^*(\xi \lrcorner \omega) = -d^\nabla \tilde{a}$ . But then  $\nabla_\rho \tilde{a} = \rho \lrcorner d^\nabla \tilde{a} = -\rho \lrcorner \pi^*(\xi \lrcorner \omega) = 0$ . Thus  $\nabla_\rho \tilde{a} = 0$ . This implies that  $\tilde{a}$  is basic, so  $\tilde{a} = \pi^*a$  for some bundle map  $a: E \rightarrow F$ . Thus,  $\mathcal{L}_{\tilde{\xi}}^\nabla \theta = 0$  if and only if there exists a bundle map  $a: E \rightarrow F$  with

$$\xi \lrcorner \omega = -d^\nabla a, \quad (3.9)$$

which says  $\xi \lrcorner \omega$  is  $d^\nabla$ -exact. Conversely, given (3.9), we immediately get  $\mathcal{L}_{\tilde{\xi}}^\nabla \omega = d^\nabla(\xi \lrcorner \omega) + \xi \lrcorner d^\nabla \omega = 0$ , since  $(d^\nabla)^2 = 0$  and  $d^\nabla \omega = 0$  and may construct  $\tilde{\xi}$  via (3.8) with  $\mathcal{L}_{\tilde{\xi}}^\nabla \theta = 0$ .

Now we compute  $[\tilde{\xi} \tilde{e}_1, \tilde{\xi} \tilde{e}_2]$  for two sections  $\tilde{e}_1, \tilde{e}_2 \in \Gamma(\pi^*E)$ . It suffices to consider  $\tilde{e}_i = \pi^*e_i$ ,  $i = 1, 2$ , for  $e_1, e_2 \in \Gamma(E)$ . Denote by  $X^\sim \in \mathfrak{X}(p)$  the horizontal lift of a vector field  $X \in \mathfrak{X}(M)$  and observe that  $\tilde{\xi}(\pi^*e_i) = (\xi e_i)^\sim$  for  $i = 1, 2$ . Hence,

$$\begin{aligned} [\tilde{\xi}(\pi^*e_1), \tilde{\xi}(\pi^*e_2)] &= [(\xi e_1)^\sim, (\xi e_2)^\sim] + [\rho\pi^*(ae_1), (\xi e_2)^\sim] \\ &\quad + [(\xi e_1)^\sim, \rho\pi^*(ae_2)] + [\rho\pi^*(ae_1), \rho\pi^*(ae_2)]. \end{aligned}$$

Thus, using (3.1), the horizontal part of  $[\tilde{\xi}(\pi^*e_1), \tilde{\xi}(\pi^*e_2)]$  is equal to

$$[\xi(e_1), \xi(e_2)]^\sim = (\xi(\nabla_{\xi e_1} e_2 - \nabla_{\xi e_2} e_1))^\sim = \tilde{\xi}(\pi^*(\nabla_{\xi e_1} e_2 - \nabla_{\xi e_2} e_1)).$$

To compute the vertical part, we consider

$$\begin{aligned} \theta([\tilde{\xi}\pi^*e_1, \tilde{\xi}\pi^*e_2]) &= -d^\nabla \theta(\tilde{\xi}\pi^*e_1, \tilde{\xi}\pi^*e_2) + \nabla_{\tilde{\xi}\pi^*e_1}(\theta(\tilde{\xi}\pi^*e_2)) - \nabla_{\tilde{\xi}\pi^*e_2}(\theta(\tilde{\xi}\pi^*e_1)) \\ &= -(\pi^*\omega)(\tilde{\xi}\pi^*e_1, \tilde{\xi}\pi^*e_2) = -\pi^*(\omega(\xi e_1, \xi e_2)) \end{aligned}$$

and

$$\begin{aligned} \theta([\rho\pi^*(ae_1), \tilde{\xi}\pi^*e_2]) &= -d^\nabla \theta(\rho\pi^*(ae_1), \tilde{\xi}\pi^*e_2) - \nabla_{\tilde{\xi}\pi^*e_2}(\theta\rho\pi^*(ae_1)) \\ &= -(\pi^*\omega)(\rho\pi^*(ae_1), \tilde{\xi}\pi^*e_2) - \nabla_{\tilde{\xi}\pi^*e_2}(\pi^*(ae_1)) \\ &= -\pi^*(\nabla_{\xi e_2}(ae_1)) = -\pi^*((\nabla_{\xi e_2} a)e_1 + a(\nabla_{\xi e_2} e_1)) \\ &= -\pi^*((d^\nabla a)(\xi e_2)e_1 + a(\nabla_{\xi e_2} e_1)) \\ &= \pi^*(\omega(\xi e_1, \xi e_2) - a(\nabla_{\xi e_2} e_1)). \end{aligned}$$

Similarly,  $\theta([\tilde{\xi}\pi^*e_1, \rho\pi^*(ae_2)]) = \pi^*(\omega(\xi e_1, \xi e_2) + a(\nabla_{\xi e_1} e_2))$ . Furthermore,

$$\begin{aligned} \theta([\rho\pi^*(ae_1), \rho\pi^*(ae_2)]) &= \nabla_{\rho\pi^*(ae_1)}(\pi^*(ae_2)) - \nabla_{\rho\pi^*(ae_2)}(\pi^*(ae_1)) \\ &= \pi^*(\nabla_{d\pi(\rho\pi^*(ae_1))}(ae_2) - \nabla_{d\pi(\rho\pi^*(ae_2))}(ae_1)) \\ &= 0 \end{aligned}$$

as  $d\pi \circ \rho = 0$ . Now  $\rho \circ \theta$  is the projection onto the vertical part and so

$$\begin{aligned} [\tilde{\xi}\pi^*e_1, \tilde{\xi}\pi^*e_2] &= \tilde{\xi}\pi^*(\nabla_{\xi e_1} e_2 - \nabla_{\xi e_2} e_1) + \rho\pi^*(\omega(\xi e_1, \xi e_2)) \\ &\quad + \rho\pi^*(a(\nabla_{\xi e_1} e_2 - \nabla_{\xi e_2} e_1)) \\ &= \tilde{\xi}(\nabla_{\tilde{\xi}\pi^*e_1}(\pi^*e_2) - \nabla_{\tilde{\xi}\pi^*e_2}(\pi^*e_1)) + \rho\pi^*(\xi^*\omega)(e_1, e_2) \\ &\quad + (\rho \circ \tilde{a})(\nabla_{\tilde{\xi}\pi^*e_1}(\pi^*e_2) - \nabla_{\tilde{\xi}\pi^*e_2}(\pi^*e_1)) \\ &= \tilde{\xi}(\nabla_{\tilde{\xi}\pi^*e_1}(\pi^*e_2) - \nabla_{\tilde{\xi}\pi^*e_2}(\pi^*e_1)) + \rho\pi^*(\xi^*\omega)(e_1, e_2). \end{aligned}$$

As  $\rho$  and  $\pi^*$  are injective, we have  $\check{\xi}(\nabla_{\check{\xi}\tilde{e}_1}\tilde{e}_2 - \nabla_{\check{\xi}\tilde{e}_2}\tilde{e}_1) = [\check{\xi}\tilde{e}_1, \check{\xi}\tilde{e}_2]$  for all  $\tilde{e}_1, \tilde{e}_2 \in \Gamma(\pi^*E)$  if and only if  $\xi^*\omega = 0$ .  $\square$

As noted in the proof, if  $\xi \lrcorner \omega$  is  $d^\nabla$ -exact and  $\omega$  is  $d^\nabla$ -closed, then we automatically get  $\mathcal{L}_\xi^\nabla \omega = 0$ . Hence, Theorem 3.5 naturally leads to the following definition.

**Definition 3.6.** *Shear data* on a smooth manifold  $M$  is a triple  $(\xi, a, \omega)$  consisting of a bundle map  $\xi: E \rightarrow TM$ , an invertible bundle morphism  $a: E \rightarrow F$  and a two-form  $\omega \in \Omega^2(M, F)$  with values in  $F$ , where  $E$  and  $F$  are flat vector bundles over  $M$  of the same rank and

- (i)  $(\xi, \nabla)$  is torsion-free (3.1),
- (ii)  $d^\nabla \omega = 0$ ,
- (iii)  $\xi \lrcorner \omega = -d^\nabla a$  and
- (iv)  $\xi^* \omega = 0$ .

Suppose additionally there is a shear total space  $(P, \theta, \rho)$  as in Definition 3.1. Define  $\check{\xi}: \pi^*E \rightarrow TP$  by  $\check{\xi} := \tilde{\xi} + \rho \circ \pi^*a$ , with  $\tilde{\xi}$  the horizontal lift of  $\xi$ . Then, by Theorem 3.5,  $\check{\xi}(\pi^*E)$  is an integrable distribution. Furthermore, invertibility of  $a$  ensures that it is of constant rank. The leaf space

$$S := P/\check{\xi}(\pi^*E)$$

is called *the shear of  $(M, \xi, a, \omega)$*  when it is a smooth manifold.

*Remark 3.7.* Locally the shear is essentially the twist construction. Given shear data  $(\xi, a, \omega)$  on a manifold  $M$ , choose parallel frames  $(e_1, \dots, e_k)$  and  $(f_1, \dots, f_k)$  of  $E$  and  $F$  locally. Then we may write  $\omega = \sum_{i=1}^k \omega_i f_i$  and  $a = \sum_{i,j=1}^k a_i^j e^i \otimes f_j$ . Putting  $\mathbf{a}_M = \mathbf{a}_P := \mathbb{R}^k$ ,  $\Omega := (\omega_1, \dots, \omega_k) \in \Omega^2(M, \mathbf{a}_P)$ ,  $A := (a_i^j)_{i,j=1,\dots,k} \in C^\infty(M, \mathbf{a}_P \otimes \mathbf{a}_M^*)$  and  $\Xi: \mathbf{a}_M \rightarrow \mathfrak{X}(M)$ ,  $\Xi(x^1, \dots, x^m) := \sum_{i=1}^k x^i \xi(e_i)$ , conditions (i)–(iv) of Definition 3.6 show that  $(\Xi, \Omega, A)$  is local twist data. Note that these identifications depend on the choices of the flat structures.

*Remark 3.8.* In some cases we may construct a suitable space  $P$  via a principal bundle on the universal cover  $\tilde{M}$  of  $M$ . The pull-back of  $F$  to  $\tilde{M}$  is flat and has a global basis of parallel sections. The pull-back of  $\omega$  may then be interpreted as the curvature two-form for a principal bundle  $\tilde{P} \rightarrow \tilde{M}$  with Abelian structure group. This imposes integrality conditions on the pull-back of  $\omega$ , and one then needs to investigate whether  $\tilde{P}$  can be chosen so that it descends to a bundle over  $M$ . We examined these questions in detail for one-dimensional fibres in [FS16]. Consideration of similar questions phrased in the language of Lie algebroids may be found in [Mac05, Mac87]. However, as will see later, candidate spaces  $P$  may arise in other ways, unrelated to principal bundles.

**3.2. Differential forms.** Let us now fix some shear data  $(\xi, a, \omega)$  on a manifold  $M$  and a corresponding triple  $(\pi: P \rightarrow M, \theta, \rho)$  as in Definition 3.6 such that the shear  $S = P/\check{\xi}(\pi^*E)$  is smooth. Write  $\pi_S: P \rightarrow S$  for the projection to  $S$ . We are interested in how to move geometric structures on  $M$  to  $S$ . In particular, how to relate  $(p, 0)$ -tensor fields on  $M$  with  $(p, 0)$ -tensor fields on  $S$ .

**Definition 3.9.** Let  $\alpha$  be a  $(p, 0)$ -tensor field on  $M$  and  $\alpha_S$  be a  $(p, 0)$ -tensor field on  $S$ . We say that  $\alpha$  is  $\mathcal{H}$ -related to  $\alpha_S$ , in symbols

$$\alpha \sim_{\mathcal{H}} \alpha_S,$$

if

$$\pi^* \alpha|_{\mathcal{H}} = \pi_S^* \alpha_S|_{\mathcal{H}}.$$

For differential forms, this relation may be concretely described.

**Proposition 3.10.** *A  $k$ -form  $\alpha$  on  $M$  is  $\mathcal{H}$ -related to some  $k$ -form  $\alpha_S$  on the shear  $S$  if and only if  $\mathcal{L}_\xi^\nabla \alpha = 0$ . In this case, the  $k$ -form  $\alpha_S$  is uniquely determined by*

$$\pi_S^* \alpha_S = \pi^* \alpha + \sum_{i=1}^k (-1)^{i(2k+1-i)/2} \pi^* ((\xi \circ a^{-1} \lrcorner)^i \alpha) \wedge \theta^i.$$

*Proof.* Suppose that  $\alpha \in \Omega^k M$  is  $\mathcal{H}$ -related to  $\alpha_S \in \Omega^k S$ . If we decompose  $\pi_S^* \alpha_S$  with respect to  $\mathcal{H}$  and  $\mathcal{V}$ , we get by definition

$$\pi_S^* \alpha_S = \pi^* \alpha + \sum_{i=1}^k \beta_i \wedge \theta^i$$

for certain  $\beta_i \in \Omega^{k-i}(P, \Lambda^i(\pi^* F)^*)$  with  $\beta_i|_{\mathcal{V}} = 0$ . Here  $\theta^i$  denotes the element  $\Lambda^i \theta \in \Gamma(\text{End}(\Lambda^i TP, \Lambda^i \pi^* F)) \cong \Omega^i(P, \Lambda^i \pi^* F)$  pointwise induced by  $\theta \in \Omega^1(P, \pi^* F) \cong \Gamma(\text{End}(TP, \pi^* F))$ . For  $X_1, \dots, X_i \in \mathfrak{X}(P)$ , this means  $\theta^i(X_1 \wedge \dots \wedge X_i) = \theta(X_1) \wedge \dots \wedge \theta(X_i)$ .

As  $\xi \lrcorner \pi_S^* \alpha_S = 0$ , we get

$$0 = \pi^*(\xi \lrcorner \alpha) + \sum_{i=1}^k (\tilde{\xi} \lrcorner \beta_i) \wedge \theta^i + \sum_{i=1}^k (-1)^{k-i} \beta_i \wedge ((\rho \circ \pi^* a) \lrcorner \theta^i)$$

As  $(\rho \circ \pi^* a) \lrcorner \theta^i = \pi^* a \theta^{i-1}$ , this tells us  $\pi^*(\xi \lrcorner \alpha) = (-1)^k \beta_1 \pi^* a$  and  $(\tilde{\xi} \lrcorner \beta_{i-1}) \wedge \theta^{i-1} = (-1)^{k-i+1} \beta_i \wedge \pi^* a \theta^{i-1}$  for  $i = 2, \dots, k$ . It follows that

$$\beta_1 = (-1)^k \pi^* ((\xi \circ a^{-1}) \lrcorner \alpha)$$

and

$$\begin{aligned} & \beta_i(X_1, \dots, X_{k-i})(\pi^* a(e), f_1, \dots, f_{i-1}) \\ &= (-1)^{k-i+1} \beta_{i-1}(\tilde{\xi}(e), X_1, \dots, X_{k-i})(f_1, \dots, f_{i-1}) \end{aligned}$$

for  $e \in \Gamma(\pi^* E)$ ,  $X_1, \dots, X_{k-i} \in \mathfrak{X}(P)$ ,  $f_1, \dots, f_{i-1} \in \Gamma(\pi^* F)$  and  $i = 2, \dots, k$ . As  $a$  is invertible, we may write  $e = \pi^* a^{-1}(f_0)$  and see that the last condition says

$$\beta_i = (-1)^{k-i+1} (\tilde{\xi} \circ \pi^* a^{-1}) \lrcorner \beta_{i-1},$$

for  $i = 2, \dots, k$ . Note that there is also the implicit statement that both sides are anti-symmetric when evaluated on  $i$  sections of  $\pi^* F$ . Recursion now gives

$$\beta_i = (-1)^{i(2k+1-i)/2} \pi^* ((\xi \circ a^{-1})^i \lrcorner \alpha)$$

and this is, in fact, anti-symmetric in sections of  $\pi^* F$ . Thus the claimed formula for  $\alpha_S$  holds, when  $\alpha_S$  exists.

So, to finish the proof, we have to consider the semi-basic  $k$ -form

$$\hat{\alpha} := \pi^* \alpha + \sum_{i=1}^k (-1)^{i(2k+1-i)/2} \pi^* ((\xi \circ a^{-1})^i \lrcorner \alpha) \wedge \theta^i$$

and have to determine when this is basic for  $\pi_S$  as exactly then  $\hat{\alpha}$  will be the pull-back of a unique  $\alpha_S$  on  $S$ . For  $\hat{\alpha}$  to be basic requires  $\mathcal{L}_{\xi_e} \hat{\alpha} = 0$  for each  $e \in \Gamma(\pi^* E)$ . This condition is

$$0 = \mathcal{L}_{\xi_e} \hat{\alpha} = \xi_e \lrcorner d\hat{\alpha} = \xi_e \lrcorner d\hat{\alpha} + d^\nabla(\xi \lrcorner \hat{\alpha})e = (\mathcal{L}_\xi^\nabla \hat{\alpha})e,$$

which just says  $\mathcal{L}_\xi^\nabla \hat{\alpha} = 0$ . Now  $\mathcal{L}_\xi^\nabla \theta = 0$ , so we have

$$\mathcal{L}_\xi^\nabla \hat{\alpha} = \pi^*(\mathcal{L}_\xi^\nabla \alpha) + \sum_{i=1}^k (-1)^{i(2k+1-i)/2} \pi^* \left( \mathcal{L}_\xi^\nabla ((\xi \circ a^{-1})^i \lrcorner \alpha) \right) \wedge \theta^i.$$

Taking the horizontal part, we see that  $\mathcal{L}_\xi^\nabla \hat{\alpha} = 0$  implies  $\mathcal{L}_\xi^\nabla \alpha = 0$ .

Conversely, assume that  $\mathcal{L}_\xi^\nabla \alpha = 0$ . For an  $\ell$ -form  $\tau$  with values in  $(F^*)^{\otimes r}$  and local parallel sections  $e$  of  $E$ ,  $f_j$  of  $F$ , note  $(\mathcal{L}_\xi^\nabla \tau)(e, f_1, \dots, f_r) = \mathcal{L}_{\xi e}(\tau(f_1, \dots, f_r))$ . Writing  $e_i := a^{-1}(f_i)$ ,  $\xi(i) = \xi(e_i)$  and  $\iota_X = X \lrcorner$ , we find

$$\begin{aligned}
& (\mathcal{L}_\xi^\nabla ((\xi \circ a^{-1} \lrcorner)^i \alpha))(e, f_1, \dots, f_i) \\
&= \mathcal{L}_{\xi e}(\iota_{\xi(i)} \dots \iota_{\xi(1)} \alpha) \\
&= \iota_{\xi(i)} \mathcal{L}_{\xi e}(\iota_{\xi(i-1)} \dots \iota_{\xi(1)} \alpha) + \iota_{[\xi e, \xi(i)]} \iota_{\xi(i-1)} \dots \iota_{\xi(1)} \alpha \\
&= \iota_{\xi(i)} \dots \iota_{\xi(1)} \mathcal{L}_{\xi e} \alpha + \sum_{j=1}^i (-1)^{j-1} \iota_{\xi(i)} \dots \widehat{\iota_{\xi(j)}} \dots \iota_{\xi(1)} \iota_{[\xi e, \xi(j)]} \alpha \\
&= \iota_{\xi(i)} \dots \iota_{\xi(1)} (\mathcal{L}_\xi^\nabla \alpha) e + \sum_{j=1}^i (-1)^{j-1} \iota_{\xi(i)} \dots \widehat{\iota_{\xi(j)}} \dots \iota_{\xi(1)} \iota_{\xi(\nabla_{\xi e} e_j - \nabla_{\xi(j)} e)} \alpha \\
&= \sum_{j=1}^i (-1)^{j-1} \iota_{\xi(i)} \dots \widehat{\iota_{\xi(j)}} \dots \iota_{\xi(1)} \iota_{\xi(\nabla_{\xi e} (a^{-1} f_j))} \alpha
\end{aligned}$$

using the torsion-free condition (3.1) in the penultimate step. But we compute

$$\begin{aligned}
\nabla_{\xi e} (a^{-1} f_j) &= (\nabla_{\xi e} a^{-1}) f_j = -(a^{-1} \circ \nabla_{\xi e} a \circ a^{-1}) f_j \\
&= a^{-1} ((\xi \lrcorner \omega)(\xi e)(e_j)) = a^{-1} (\omega(\xi e_j, \xi e)) = 0,
\end{aligned}$$

and so  $(\mathcal{L}_\xi^\nabla ((\xi \circ a^{-1} \lrcorner)^i \alpha))(e, f_1, \dots, f_i) = 0$ . As the local parallel sections span  $E$  and  $F$ , we get  $\mathcal{L}_\xi^\nabla ((\xi \circ a^{-1} \lrcorner)^i \alpha) = 0$  and conclude that  $\mathcal{L}_\xi^\nabla \hat{\alpha} = 0$ .  $\square$

Next we consider the differentials of  $\mathcal{H}$ -related  $k$ -forms:

**Corollary 3.11.** *Let  $\alpha \in \Omega^k M$  be a  $k$ -form on  $M$  with  $\mathcal{L}_\xi^\nabla \alpha = 0$  and take  $\alpha_S \in \Omega^k S$  with  $\alpha \sim_{\mathcal{H}} \alpha_S$ . Then*

$$d\alpha - (\xi \circ a^{-1} \lrcorner \alpha) \wedge \omega \sim_{\mathcal{H}} d\alpha_S.$$

*Proof.* As  $\mathcal{L}_\xi^\nabla \alpha = 0$ ,  $\mathcal{L}_\xi^\nabla \omega = 0$  and  $\mathcal{L}_\xi^\nabla (\xi \circ a^{-1} \lrcorner \alpha) = 0$  by the proof of Proposition 3.10, we have  $\mathcal{L}_\xi^\nabla (d\alpha - (\xi \circ a^{-1} \lrcorner \alpha) \wedge \omega) = 0$ . So Proposition 3.10 tells us that there exists a unique  $(k+1)$ -form on  $S$  which is  $\mathcal{H}$ -related to  $d\alpha - (\xi \circ a^{-1} \lrcorner \alpha) \wedge \omega$ . This  $(k+1)$ -form has to be equal to  $d\alpha_S$  as differentiating the formula for  $\pi_S^* \alpha_S$  in Proposition 3.10 gives us horizontally

$$\pi_S^* d\alpha_S = \pi^* d\alpha - \pi^* (\xi \circ a^{-1} \lrcorner \alpha) \wedge d^\nabla \theta = \pi^* (d\alpha - (\xi \circ a^{-1} \lrcorner \alpha) \wedge \omega),$$

as claimed.  $\square$

**3.3. Vector fields and almost complex structures.** With the notation from the previous section, we say that a vector field  $X$  on  $M$  is  $\mathcal{H}$ -related to a vector field  $X_S$  on  $S$  if and only if their horizontal lifts  $\tilde{X}$ ,  $\widehat{X_S}$  to  $P$  agree.

For given  $X \in \mathfrak{X}(M)$  there is at most one  $X_S \in \mathfrak{X}(S)$  with  $X \sim_{\mathcal{H}} X_S$ . Furthermore,  $X_S$  exists if and only if  $[\tilde{X}, \Gamma(\dot{\xi}(\pi^* E))] \subset \Gamma(\dot{\xi}(\pi^* E))$ . Now from equation (3.6) we obtain that for given  $e \in \Gamma(E)$  the vertical component of  $[\tilde{X}, \dot{\xi}(\pi^* e)]$  equals the vertical component of  $\dot{\xi}(\nabla_{\tilde{X}} \pi^* e) = \dot{\xi}(\pi^*(\nabla_X e))$  and the horizontal component of  $[\tilde{X}, \dot{\xi}(\pi^* e)]$  equals  $[X, \xi(e)]$ . So the existence of  $X_S \in \mathfrak{X}(S)$  with  $X \sim_{\mathcal{H}} X_S$  is equivalent to  $[\xi(e), X] = -\xi(\nabla_X e)$  for all  $e \in \Gamma(E)$ . Defining  $(\mathcal{L}_\xi^\nabla X)(e) := [\xi(e), X] + \xi(\nabla_X e)$  for  $e \in \Gamma(E)$ , the  $\mathcal{H}$ -related vector field  $X_S$  exists if and only if

$$\mathcal{L}_\xi^\nabla X = 0.$$

Note that  $\mathcal{L}_\xi^\nabla$  behaves as the usual Lie derivative under contractions:  $(\mathcal{L}_\xi^\nabla \alpha)(X) = \mathcal{L}_\xi^\nabla (\alpha(X)) - \alpha(\mathcal{L}_\xi^\nabla X)$  for all  $X \in \mathfrak{X}(M)$  and all  $\alpha \in \Omega^1 M$ .

After these preliminaries, we consider Lie brackets of  $\mathcal{H}$ -related vector fields:

**Lemma 3.12.** *Suppose the vector fields  $X, Y \in \mathfrak{X}(M)$ ,  $X_S, Y_S \in \mathfrak{X}(S)$  satisfy  $X \sim_{\mathcal{H}} X_S$  and  $Y \sim_{\mathcal{H}} Y_S$ . Then the identity*

$$[X, Y] + \xi a^{-1} \omega(X, Y) \sim_{\mathcal{H}} [X_S, Y_S]$$

*holds.*

*Proof.* Considering the decomposition into horizontal and vertical subspaces of  $\pi$ , we have  $[\tilde{X}, \tilde{Y}] = [\widehat{X}, \widehat{Y}] + \rho\theta[\tilde{X}, \tilde{Y}]$ . For  $\pi_S$ , note that the projection to the vertical subspace of  $\pi_S$  is given by  $\xi\pi^*a^{-1}\theta = \tilde{\xi}\pi^*a^{-1}\theta + \rho\theta$ . This gives

$$\begin{aligned} [\tilde{X}, \tilde{Y}] &= [\widehat{X}_S, \widehat{Y}_S] = [\widehat{X}_S, \widehat{Y}_S] + \tilde{\xi}\pi^*a^{-1}\theta[\widehat{X}_S, \widehat{Y}_S] + \rho\theta[\widehat{X}_S, \widehat{Y}_S] \\ &= [\widehat{X}_S, \widehat{Y}_S] - \tilde{\xi}\pi^*(a^{-1}\omega(X, Y)) + [\tilde{X}, \tilde{Y}] - [\widehat{X}, \widehat{Y}]. \end{aligned}$$

Thus  $([X, Y] + \xi a^{-1} \omega(X, Y))^\sim = [\widehat{X}_S, \widehat{Y}_S]$  as claimed.  $\square$

Next, we consider almost complex structures  $I$  on  $M$  and  $I_S$  on  $S$  and say that they are  $\mathcal{H}$ -related if and only if for all  $p \in P$  and all  $v \in T_{\pi(p)}M$  we have  $\tilde{I}v = (I_S d\pi_S \tilde{v})^\wedge$ . Then, given an almost complex structure  $I$  on  $M$ , there exists an  $\mathcal{H}$ -related almost complex structure on  $S$  if and only if  $\mathcal{L}_\xi^\nabla I = 0$ , where we extend  $\mathcal{L}_\xi^\nabla$  in the usual way to  $(1,1)$ -tensors. Moreover, the definition of the Nijenhuis tensor  $N_I$  of an almost complex structure  $I$  and Lemma 3.12 yield:

**Proposition 3.13.** *For almost complex structures  $I$  on  $M$  and  $I_S$  on  $S$  with  $I \sim_{\mathcal{H}} I_S$ , the Nijenhuis tensors are related by*

$$N_I - \mathcal{F} - I\mathcal{F}(I \cdot, \cdot) - I\mathcal{F}(\cdot, I \cdot) + \mathcal{F}(I \cdot, I \cdot) \sim_{\mathcal{H}} N_{I_S},$$

where  $\mathcal{F} = \xi a^{-1} \omega$ .  $\square$

**3.4. Duality.** In this section, we assume that we are in the situation of §3.1 and show how we can then invert the shear construction.

A first necessary condition to invert the shear is to construct flat vector bundles  $E_S$  and  $F_S$  over the shear  $S$  such that  $\pi_S^* E_S \cong \pi^* F$  and  $\pi_S^* F_S \cong \pi^* E$  as bundles with flat connections. If we have such vector bundles and if we fix identifications  $\pi_S^* E_S \cong \pi^* F$  and  $\pi_S^* F_S \cong \pi^* E$ , we may extend  $\sim_{\mathcal{H}}$  to  $k$ -forms and vector fields with values in tensor powers of these bundles. For example, we say that  $\alpha \in \Omega^k(M, E^{\otimes r} \otimes F^{\otimes s})$  is  $\mathcal{H}$ -related to  $\alpha_S \in \Omega^k(S, E_S^{\otimes r} \otimes F_S^{\otimes s})$  if and only if  $\pi^* \alpha|_{\mathcal{H}} = \pi_S^* \alpha_S|_{\mathcal{H}}$ .

**Theorem 3.14.** *Suppose  $(M, E, \xi, F, a, \omega)$  shears via a total space  $(P, \theta, \rho)$  to  $S$ . Suppose in addition that  $\pi^* E$  and  $\pi^* F$  can be trivialised by flat sections in a neighbourhood of each leaf of  $\xi$  on  $P$ . Then there is shear data  $(\xi_S, a_S, \omega_S)$  on  $S$  realising  $M$  as the shear of  $S$ .*

*More precisely, there are flat bundles  $E_S, F_S \rightarrow S$ , a torsion-free bundle map  $\xi_S: E_S \rightarrow TS$ , a two-form  $\omega_S \in \Omega^2(S, F_S)$  and an  $a_S \in \Omega^0(S, E_S^* \otimes F_S)$  such that*

- (a) *as flat bundles  $\pi_S^* E_S \cong \pi^* F$  and  $\pi_S^* F_S \cong \pi^* E$ ,*
- (b)  *$a^{-1} \sim_{\mathcal{H}} a_S$ ,  $a^{-1} \omega \sim_{\mathcal{H}} \omega_S$ ,  $-\xi \circ a^{-1} \sim_{\mathcal{H}} \xi_S$ ,*
- (c)  *$M$  is the shear of  $S$  via  $(P, \theta_S = (\pi^* a^{-1})\theta, \rho_S = \xi)$ .*

*Proof.* Recall that the fibres of  $\pi_S: P \rightarrow S$  are the leaves of  $\xi$  and that by assumption  $\pi^* F$  can be trivialised by flat sections in a neighbourhood of each leaf. So we may define a locally free sheaf  $\mathcal{F}$  on  $S$  by letting  $\mathcal{F}(U)$  be those sections  $\sigma$  of  $\pi^* F$  over  $\pi_S^{-1}(U)$  which are constant on the leaves under a trivialisation by flat sections.

Let  $E_S$  be the associated vector bundle; by construction we have  $\pi_S^* E_S = \pi^* F$ . We give  $E_S$  a flat connection  $\nabla$  by declaring a local section to be parallel if its

pull-back to  $\pi^*F$  is parallel. In a similar way, we construct a flat bundle  $F_S \rightarrow S$  with  $\pi_S^*F_S = \pi^*E$ .

Computing

$$\mathcal{L}_{\dot{\xi}}^\nabla \dot{a} = \dot{\xi} \lrcorner d^\nabla(\dot{a}) = -\dot{\xi} \lrcorner \pi^*(\xi \lrcorner \omega) = -\pi^*(\xi^* \omega) = 0,$$

we conclude that  $\dot{a} = \pi_S^* a_S^{-1}$  for some vector bundle morphism  $a_S: E_S \rightarrow F_S$ .

To define  $\xi_S: E_S \rightarrow TS$ , we wish to push  $\rho: \pi_S^*E_S = \pi^*F \rightarrow TP$  forward so that the diagram

$$\begin{array}{ccc} \pi_S^*E_S & \xrightarrow{\rho} & TP \\ \downarrow & & \downarrow d\pi_S \\ E_S & \xrightarrow{\xi_S} & TS. \end{array}$$

commutes. We thus define  $(\xi_S)_x(e_x) := (d\pi_S)_p(\rho_p(p, e_x))$  for any  $x \in S$ ,  $e_x \in (E_S)_x$  and for some  $p \in \pi_S^{-1}(x)$ , but need to show this is independent of  $p$ . To this end, let  $e$  be a local parallel section of  $E_S$  extending  $e_x$ . Then  $\theta\rho(\pi_S^*e) = \pi_S^*\dot{e}$  is parallel, so (3.7) gives that  $[\rho(\pi_S^*e), \dot{\xi}(\pi^*e)] = 0$  for each local parallel section  $e$  of  $E$ . Thus  $\rho(\pi_S^*e)$  is projectable, as required, and  $\xi_S$  is well-defined.

For the torsion-free condition, let  $e_1$  and  $e_2$  be two sections of  $E_S$ . Then  $\rho(\pi_S^*e_i)$  projects to  $\xi_S e_i$ , so  $[\rho(\pi_S^*e_1), \rho(\pi_S^*e_2)]$  projects to  $[\xi_S e_1, \xi_S e_2]$ . Moreover,

$$\nabla_{\rho(\pi_S^*e_1)} \pi_S^*e_2 - \nabla_{\rho(\pi_S^*e_2)} \pi_S^*e_1 = \pi_S^* (\nabla_{\xi_S(e_1)} e_2 - \nabla_{\xi_S(e_2)} e_1)$$

and so  $\rho(\nabla_{\rho(\pi_S^*e_1)} \pi_S^*e_2 - \nabla_{\rho(\pi_S^*e_2)} \pi_S^*e_1)$  projects to  $\xi_S(\nabla_{\xi_S(e_1)} e_2 - \nabla_{\xi_S(e_2)} e_1)$ . Thus,  $\xi_S$  is torsion-free as  $\rho$  is torsion-free by Remark 3.4.

Defining  $\theta_S := \dot{a}^{-1} \circ \theta \in \Omega^1(P, \pi_S^*F_S)$  and  $\rho_S := \dot{\xi}$ , we verify the conditions (1)–(3) of Definition 3.1. Firstly,  $\theta_S \circ \rho_S = \dot{a}^{-1} \circ \theta \circ (\tilde{\xi} + \rho \circ \dot{a}) = \dot{a}^{-1} \theta \rho \dot{a} = \text{id}_{\pi^*E} = \text{id}_{\pi_S^*F_S}$ . Furthermore,  $S$  is defined to be the leaf space of  $\dot{\xi}$ , so  $d\pi_S \circ \dot{\xi} = 0$ . Now a computation gives

$$\begin{aligned} d^\nabla \theta_S &= d^\nabla \dot{a}^{-1} \wedge \theta + \dot{a}^{-1} d^\nabla \theta = \pi^*(a^{-1} \xi \lrcorner \omega a^{-1}) \wedge \theta + \pi^*(a^{-1} \omega) \\ &= \pi^*(a^{-1} \xi \lrcorner \omega) \wedge \theta_S + \pi^*(a^{-1} \omega). \end{aligned}$$

From this we conclude  $\dot{\xi} \lrcorner d^\nabla \theta_S = \pi^*(a^{-1} \xi^* \omega) \wedge \theta_S = 0$  and so  $\mathcal{L}_{\dot{\xi}}^\nabla d^\nabla \theta_S = 0$ . Thus, there is a two-form  $\omega_S \in \Omega^2(S, F_S)$  with  $\pi_S^* \omega_S = d^\nabla \theta_S$ . Indeed  $a^{-1} \omega \sim_{\mathcal{H}} \omega_S$ .

Having found  $a_S$  and  $\omega_S$ , it remains to verify (ii)–(iv) in Definition 3.6. (ii) is automatic from the flatness of  $\nabla$ . We compute

$$\begin{aligned} \pi_S^* d^\nabla a_S &= d^\nabla \dot{a}^{-1} = -\dot{a}^{-1} d^\nabla \dot{a} \dot{a}^{-1} = \pi^*(a^{-1} \xi \lrcorner \omega a^{-1}) \\ &= -\rho \lrcorner d^\nabla \theta_S = -\pi_S^*(\xi_S \lrcorner \omega_S), \end{aligned}$$

giving (iii). The final condition follows from

$$\pi_S^*(\xi_S^* \omega_S) = \rho^*(\pi_S^* \omega_S) = \rho^* d^\nabla \theta_S = \rho^*(\pi^*(a^{-1} \xi \lrcorner \omega) \wedge \theta_S + \pi^*(a^{-1} \omega)) = 0.$$

Thus we have some shear data  $(\xi_S, \omega_S, a_S)$  on  $S$ .

To see that we obtain  $M$  as the resulting shear, it is sufficient to show that  $\dot{\xi}_S = \rho$ . But  $\dot{\xi} = \tilde{\xi} + \rho \circ \dot{a}$  and  $\pi_S^* a_S = \dot{a}^{-1}$  imply

$$\rho = -\tilde{\xi} \circ \dot{a}^{-1} + \dot{\xi} \circ \dot{a}^{-1} = -\tilde{\xi} \circ \dot{a}^{-1} + \rho_S \circ \pi_S^* a_S.$$

Now the final term is  $\pi_S$ -vertical and the first one lies in  $\mathcal{H} = \ker \theta = \ker \theta_S$ . As  $\rho$  projects to  $\xi_S$ , we conclude that  $\tilde{\xi}_S = -\tilde{\xi} \circ \dot{a}^{-1}$ , thus  $\dot{\xi}_S = \rho$  and  $-\xi \circ a^{-1} \sim_{\mathcal{H}} \xi_S$ .  $\square$

## 4. EXAMPLES

**4.1. Shears via non-principal bundles.** In [FS16], we considered a first version of a rank one shear construction with the bundles  $E$  and  $F$  trivialised. We showed in [FS16, §3.4] that then  $P$  always has the structure of a principal bundle. This is no longer true in the general shear construction, even when  $P$  has rank one. This fails for two reasons: we no longer require  $P$  to be a fibre bundle and we only require  $E$  and  $F$  to be flat, not trivial.

Let us give a simple example where  $P$  is not a fibre bundle and the topology of the fibres can change. Take  $P = T^2 = S^1 \times S^1$ ,  $M = S^1$  and  $W = S^1$ . There is a twist of the form  $M = S^1 \leftarrow P = T^2 \rightarrow W = S^1$  with the first map being the projection onto the first  $S^1$ -factor and the second one given by  $(x, y) \mapsto x^{-1}y$ . This corresponds to twist data  $(\xi, a, \omega)$  with Abelian Lie algebras  $\mathfrak{a}_P = \mathfrak{a}_M = \mathbb{R}$ ,  $\xi(1) = \partial_\varphi$ ,  $a = \text{id}_{\mathbb{R}}$ ,  $\omega = 0$  and  $\theta = d\psi$ , where  $\varphi$  describes the first  $S^1$ -factor and  $\psi$  the second one in  $T^2$ . Taking  $E = F = M \times \mathbb{R}$  with the natural flat connections, this can also be understood as a shear construction.

Now we may remove one point  $p \in T^2$ . Then, still  $M = S^1 \leftarrow P' = T^2 \setminus \{p\} \rightarrow S = S^1$  is a shear, but neither map  $P' \rightarrow M, S$  is a fibre bundle, indeed most fibres are circles, but one fibre is homeomorphic to  $\mathbb{R}$ . More generally, one may remove a closed segment  $L$  in one fibre of  $T^2 \rightarrow M = S^1$ , then  $P'' = T \setminus L \rightarrow S = S^1$  is still surjective but has fibres which are topologically  $\mathbb{R}$  over a closed segment in  $S = S^1$ .

Even if we assume that  $P \rightarrow M$  is a fibre bundle, we cannot conclude that  $P$  has the structure of a principal bundle. To see this, take a non-trivial flat vector bundle  $(E, \nabla)$  of rank  $k$  and  $P = F = E$ . Then  $P = E \rightarrow M$  does not admit the structure of a principal  $\mathbb{R}^k$ -bundle: if it did, the structure group could be reduced to the maximal compact subgroup  $\{0\} \leq \mathbb{R}^k$  contradicting non-triviality.

Take the Ehresmann connection  $\hat{\theta} \in \Omega^1(E, \mathcal{V}) \subset \text{End}(TE)$  associated to  $\nabla$ . We get an induced splitting  $TE = \mathcal{H} \oplus \mathcal{V} \cong \pi^*TM \oplus \pi^*E$  and  $\theta$  corresponds to the projection onto the second factor. Using Proposition 3.3, we see that  $(P, \theta, \text{inc})$  is a shear total space for  $\omega = 0$ :

- For  $X_i = \pi^*e_i$  vertical with  $e_i \in \Gamma(E)$ ,  $i = 1, 2$ , one gets  $[X_1, X_2] = 0$  as  $\varphi_t^{X_i}(\tilde{e}) = \tilde{e} + te_i(\pi(\tilde{e}))$  is the flow of  $X_i = \pi^*e_i$  and  $\nabla_{X_i}X_j = \nabla_{X_i}\pi^*e_j = \pi^*(\nabla_{d\pi(X_i)}e_j) = 0$  for all  $\{i, j\} = \{1, 2\}$ . Thus, equation (3.2) holds.
- Let  $X = \pi^*e$  for a parallel (local) section  $e \in \Gamma(E)$  and let  $Y$  be horizontal. Then  $[X, Y] = \frac{d}{dt}|_{t=0}d\varphi_{-t}^X(Y(\varphi(t)))$  for  $\varphi_t^X(\tilde{e}) = \tilde{e} + te(\pi(\tilde{e}))$ . As our Ehresmann connection comes from a covariant derivative  $\nabla$  on  $E$ , the differentials of the scalar multiplication and the addition on  $E$  preserve the horizontal subbundle. Moreover,  $de(d\pi(v))$  is horizontal for any  $v \in TE$  as  $e$  is parallel. So  $d\varphi_{-t}^X(Y(\varphi(t)))$  is in the horizontal subbundle for all  $t$ , which implies that  $X = \pi^*e$  preserves the horizontal subbundle.

Moreover, if  $\xi: E \rightarrow TM$  is any bundle map and  $a: E \rightarrow E$  any bundle isomorphism, then  $(\xi, a, 0)$  defines shear data exactly when  $(\xi, \nabla)$  is torsion-free and  $\nabla a = d^\nabla a = 0$ . For such a shear, we have  $d\alpha \sim_{\mathcal{H}} d_S \alpha_S$  if  $\alpha \sim_{\mathcal{H}} \alpha_S$ . As we may always take  $\xi = 0$  and  $a = \text{id}_E$  to get a shear  $M \leftarrow E \rightarrow M$  for which both projections coincide, there are, in fact, examples of rank one shears for which  $P \rightarrow M$  does not have the structure of a principal bundle.

Note that if  $E = TM$  and  $\xi = \text{id}_{TM}$ , then  $(\xi, \nabla)$  being torsion-free means exactly that  $\nabla$  is torsion-free. We may take then  $a = J$  to be an almost complex structure on  $M$ . As  $\nabla J = 0$  and  $\nabla$  torsion-free implies that  $J$  is integrable, a triple  $(J, \text{id}_{TM}, 0)$  defines shear data if and only if  $(M, J, \nabla)$  is a special complex manifold in the sense of [ACD01] with  $\nabla$  being complex, i.e.  $\nabla J = 0$ . If, conversely, we take  $\xi = J$  being an almost complex structure and  $a = \text{id}_{TM}$ , then  $(\text{id}_{TM}, J, 0)$  defines shear data precisely when  $[JX, JY] = J(\nabla_{JX}Y - \nabla_{JY}X)$  for all  $X, Y \in \mathfrak{X}(M)$ . If



additionally  $\nabla$  is torsion-free, this condition implies that  $J$  is integrable. Hence, for  $\nabla$  being torsion-free,  $(\text{id}_{TM}, J, 0)$  defines shear data if and only if  $(M, J, \nabla)$  is a special complex manifold.

If  $E = TM$ , we may also take  $\omega$  to be the torsion  $T^\nabla$  of a non-torsion-free flat connection  $\nabla$ . Namely, first of all, a short computation shows

$$d^\nabla T^\nabla(X, Y, Z) = R^\nabla(X, Y)Z + R^\nabla(Y, Z)X + R^\nabla(Z, X)Y = 0.$$

Furthermore, we have  $\mathcal{V} \cong \pi^*TM$  in  $TP = TTM = \pi^*TM \oplus \pi^*TM = \mathcal{H} \oplus \mathcal{V}$ . So the one-form  $A \in \Omega^1(TM, \pi^*TM)$ ,  $A := \theta + \eta$  with  $\theta$  as above and  $\eta \in \Omega^1(TM, \pi^*TM)$  being the projection onto the horizontal subbundle, satisfies  $d^\nabla A = d^\nabla \eta = \pi^*T^\nabla$ . Hence,  $(\rho, A, T^\nabla)$  is a shear total space for  $T^\nabla$ . Then a triple  $(\xi, a, T^\nabla)$  with  $\xi, a \in \text{End}(TM)$  and  $a$  being invertible defines shear data if and only if  $T^\nabla|_{\xi(TM) \times \xi(TM)} = 0$ ,  $T^\nabla(\cdot, \xi(\cdot)) = \nabla a$  and  $\nabla_\xi \xi$  is symmetric.

**4.2. Shears on Lie algebras revisited.** In §2.2, we introduced the shear construction on Lie algebras by replacing in the left-invariant twist construction central ideals and central extension by Abelian ones. The results of that section then motivated our general definition of shear data and the shear in Definition 3.6. Here, we like to see how we can recover the shear on Lie algebras from “left-invariant” shear data on a Lie group  $G$ .

**Definition 4.1.** Let  $G$  be a 1-connected Lie group and  $E$  and  $F$  be trivial vector bundles of rank  $k$  over  $G$  endowed with flat connections  $\nabla^E$  and  $\nabla^F$ , respectively, which are both *left-invariant* in the sense that the connection forms with respect to frames of constant sections are left-invariant. We write  $E = G \times \mathfrak{a}_G$  and  $F = G \times \mathfrak{a}_P$  for  $k$ -dimensional Abelian Lie algebras  $\mathfrak{a}_G$  and  $\mathfrak{a}_P$ .

Suppose  $(\xi, a, \omega) \in \Gamma(\text{Hom}(E, TG)) \times \Gamma(\text{Hom}(E, F)) \times \Omega^2(G, F)$  is shear data on  $G$ . Let  $G$  act by left translations and their differentials on  $G$  and  $TG$ , respectively, on  $E$  and  $F$  by left-translation on the first and trivially on the second factor and extend these actions naturally to tensor products of these vector bundles. Then we say the shear data is *left-invariant* if  $(\xi, a, \omega)$  are  $G$ -equivariant sections of the corresponding bundles and  $\mathcal{L}_\xi^{\nabla^E} \alpha = 0$  for all left-invariant one-forms  $\alpha \in \Omega^1 G$ .

Note that  $(\xi, a, \omega)$  being  $G$ -equivariant is equivalent to  $\xi$  mapping constant sections of  $E$  to left-invariant vector fields,  $a$  mapping constant sections of  $E$  to constant sections of  $F$  and  $\omega$  mapping pairs of left-invariant vector fields to constant sections of  $F$ . So we can consider  $a$  as an element of  $\mathfrak{a}_G^* \otimes \mathfrak{a}_P$ ,  $\xi$  as a Lie algebra homomorphism from  $\mathfrak{a}_G$  to  $\mathfrak{g}$  and  $\omega$  as an element of  $\Lambda^2 \mathfrak{g}^* \otimes \mathfrak{a}_P$ . The flat connections are determined by  $\gamma \in \mathfrak{g}^* \otimes \mathfrak{gl}(\mathfrak{a}_G)$  and  $\eta \in \mathfrak{g}^* \otimes \mathfrak{gl}(\mathfrak{a}_P)$ , so  $\nabla_X^E e = X(e) + \gamma(X)(e)$  or  $\nabla_X^F e = X(f) + \eta(X)(f)$ , for any  $X \in \mathfrak{X}(G)$  and any  $e \in \Gamma(E)$ ,  $f \in \Gamma(F)$ .

Condition (iii) for shear data in Definition 3.6 is  $\xi \lrcorner \omega = -d^\nabla a$ . As

$$(d^\nabla a)(X, Y) = (\nabla_Y a)(X) = \nabla_Y^F(aX) - a(\nabla_Y^E X) = \eta(Y)(aX) - a(\gamma(Y)(X))$$

for all  $X \in \mathfrak{a}_P$ ,  $Y \in \mathfrak{g}$ , this gives us  $\gamma = a^{-1}(\xi \lrcorner \omega) + a^{-1}\eta a$ , which is equation (2.2). Moreover,  $(\xi, \nabla)$  being torsion-free yields

$$[\xi X, \xi Y] = \xi(\nabla_{\xi X} Y - \nabla_{\xi Y} X) = \xi(\gamma(\xi X)(Y) - \gamma(\xi Y)(X)) \quad (4.1)$$

for all  $X, Y \in \mathfrak{a}_G$ . Furthermore, we have

$$\begin{aligned} 0 &= (\mathcal{L}_\xi^\nabla \alpha)(X, \xi Y) = d^\nabla(\xi \lrcorner \alpha)(X, \xi Y) + (\xi \lrcorner d\alpha)(X, \xi Y) \\ &= (\nabla_{\xi Y} \alpha \circ \xi)(X) + d\alpha(\xi X, \xi Y) = -\alpha(\xi \nabla_{\xi Y} X + [\xi X, \xi Y]) \\ &= -\alpha(\xi(\gamma(\xi Y)(X)) + [\xi X, \xi Y]) \end{aligned}$$

for all  $\alpha \in \mathfrak{g}^*$ , and so  $[\xi X, \xi Y] = -\xi(\gamma(\xi Y)(X))$  for all  $X, Y \in \mathfrak{a}_G$ . With (4.1) we deduce that  $\xi(\gamma(\xi X)(Y)) = 0$  for all  $X, Y \in \mathfrak{a}_G$  and so  $[\xi X, \xi Y] = 0$ . Hence

$\xi: \mathfrak{a}_G \rightarrow \mathfrak{g}$  is a Lie algebra homomorphism. Thus, we got the same data as in §2.2 fulfilling the requirements of Lemma 2.2, except that  $\xi$  is not necessarily injective.

Definition 3.6(ii) yields  $0 = d^\nabla \omega = d\omega + \eta \wedge \omega$ . Using the flatness of  $\nabla^F$ , we get that  $\eta \in \mathfrak{g}^* \otimes \mathfrak{gl}(\mathfrak{a}_P) = \text{Hom}(\mathfrak{g}, \mathfrak{gl}(\mathfrak{a}_P))$  is a representation.

We may define now a natural shear total space for  $\omega$  as follows: First of all, as in §2.2,  $\omega$  and  $\eta$  define an Abelian extension  $\mathfrak{a}_P \hookrightarrow \mathfrak{p} \twoheadrightarrow \mathfrak{g}$  together with a vector space splitting  $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{a}_P$ . Let  $P$  be the 1-connected Lie group with Lie algebra  $\mathfrak{p}$  and define  $\theta \in \Omega^1(P, \pi^* F) = \Omega^1(P, \mathfrak{a}_P)$  and  $\rho: P \times \mathfrak{a}_P = \pi^* F \rightarrow TP$  as in §2.2, i.e.  $\theta$  corresponds to the projection  $\mathfrak{p} \rightarrow \mathfrak{a}_P$  onto  $\mathfrak{a}_P$  induced by the splitting and  $\rho$  to the injection  $\mathfrak{a}_P \hookrightarrow \mathfrak{p}$ . Then  $(P, \theta, \rho)$  is a shear total space for  $\omega$ .

Moreover,  $\tilde{\xi}: P \times \mathfrak{a}_G = \pi^* E \rightarrow TP$ ,  $\tilde{\xi} = \tilde{\xi} + \rho \circ a$  can be considered as a linear map from  $\mathfrak{a}_G$  to  $\mathfrak{p}$  and Lemma 2.2 gives us that  $\tilde{\xi}$  is a Lie algebra homomorphism and that  $\tilde{\xi}(\mathfrak{a}_G)$  is an Abelian ideal in  $\mathfrak{p}$ . The leaves of the distribution  $\tilde{\xi}(\mathfrak{a}_G)$  are the left cosets of the normal Lie subgroup  $N$  of  $P$  with Lie algebra  $\tilde{\xi}(\mathfrak{a}_G)$ . Thus, the shear is the Lie group  $H := P/N$  with Lie algebra  $\mathfrak{h}$  agreeing with that of Definition 2.4. In this sense, the left-invariant shear on 1-connected Lie groups with injective  $\xi$  via the natural shear total space  $(P, \theta, \rho)$  from above and the shear on Lie algebras presented in §2.2 are the same and we will not distinguish them in the following sections.

**4.3. Shears on almost Abelian Lie algebras.** As explained at the end of §2.2, we can successively shear  $\mathbb{R}^n$  to any  $n$ -dimensional solvable Lie algebra such that each shear increases the solvable step length by one. One important example of solvable Lie algebras of step length one is provided by almost Abelian Lie algebras:  $\mathfrak{g}$  with a codimension one Abelian ideal  $\mathfrak{u}$ . Choosing  $X \in \mathfrak{g} \setminus \mathfrak{u}$ , these Lie algebras are determined by a single endomorphism  $f := \text{ad}(X)|_{\mathfrak{u}} \in \text{End}(\mathfrak{u})$ . Experience shows that a range of different geometric structures may be constructed on these  $\mathfrak{g}$  [Fre12, Fre13].

The following proposition gives conditions when left-invariant data  $(\xi, a, \omega) \in \text{Hom}(\mathfrak{a}_G, \mathfrak{g}) \times \text{Hom}(\mathfrak{a}_G, \mathfrak{a}_P) \times \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{a}_P$  of a particular form is shear data on the associated simply-connected Lie group  $G$  and when the shear of a closed left-invariant form is again closed. We write  $\alpha \in \mathfrak{g}^*$  for the unique element in the annihilator of  $\mathfrak{u}$  with  $\alpha(X) = 1$ .

We will consider an  $f$ -invariant subspace  $\mathfrak{a}$  of  $\mathfrak{u}$ , take  $\mathfrak{a}_G = \mathfrak{a} = \mathfrak{a}_P$  and let  $E := G \times \mathfrak{a}_G$ ,  $F := G \times \mathfrak{a}_P$  be trivial vector bundles endowed with flat left-invariant connections determined by  $\gamma \in \mathfrak{g}^* \otimes \mathfrak{gl}(\mathfrak{a}_G)$  and  $\eta \in \mathfrak{g}^* \otimes \mathfrak{gl}(\mathfrak{a}_P)$ , respectively.

**Proposition 4.2.** *Let  $G$  be a simply-connected almost Abelian Lie group with Lie algebra data  $(\mathfrak{g}, \mathfrak{u}, X, \alpha, f)$ . Let  $\mathfrak{a} \subset \mathfrak{u}$  be an  $f$ -invariant subspace and take flat bundles as above.*

*Fix a left-invariant two-form  $\omega \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{a}_P$ . Consider the decomposition  $\omega = \omega_0 + \alpha \wedge \nu$ , with  $\omega_0 \in \Lambda^2 \mathfrak{u}^* \otimes \mathfrak{a}$  and  $\nu \in \mathfrak{u}^* \otimes \mathfrak{a} \subset \text{End}(\mathfrak{u})$ .*

*Assume that  $\omega_0 \neq 0$  and  $\omega_0|_{\mathfrak{a} \otimes \mathfrak{u}} = 0$ .*

- (a) *Then  $(\text{inc}, \text{id}_{\mathfrak{a}}, \omega)$ , with  $\text{inc}: \mathfrak{a} \rightarrow \mathfrak{g}$  the inclusion, is left-invariant shear data on  $G$  if and only if*

$$f \cdot \omega_0 = -(f + \nu) \circ \omega_0, \quad \gamma = \alpha \otimes f|_{\mathfrak{a}}, \quad \eta = \alpha \otimes (f + \nu)|_{\mathfrak{a}}. \quad (4.2)$$

- (b) *Suppose  $(\text{inc}, \text{id}_{\mathfrak{a}}, \omega)$  is left-invariant shear data on  $G$ . Let  $\psi \in \Lambda^r \mathfrak{g}^*$  be a closed left-invariant  $r$ -form on  $G$  and decompose  $\psi$  uniquely as  $\psi = \chi \wedge \alpha + \tau$  with  $\chi \in \Lambda^{r-1} \mathfrak{u}^*$  and  $\tau \in \Lambda^r \mathfrak{u}^*$ . Then the  $\mathcal{H}$ -related form  $\psi_{\mathfrak{h}}$  on the shear  $\mathfrak{h}$  is closed if and only if*

$$\kappa(\tau \wedge \omega_0) = 0, \quad \kappa(\chi \wedge \omega_0) + (-1)^{r-1} \nu \cdot \tau = 0, \quad (4.3)$$

where  $\kappa: \Lambda^k \mathfrak{u}^* \otimes \mathfrak{a} \rightarrow \Lambda^{k-1} \mathfrak{u}^*$  is the unique linear map given on decomposable elements  $\rho \otimes A$  by  $\kappa(\rho \otimes A) = A \lrcorner \rho$ .

In the above statement  $(f.\omega_0)(X, Y) = -\omega_0(fX, Y) - \omega_0(X, fY)$  for  $X, Y \in \mathfrak{u}$ .

*Remark 4.3.* Here is one motivation for the case considered. As  $a: \mathfrak{a}_G \rightarrow \mathfrak{a}_P$  is invertible, it is no restriction to assume that  $\mathfrak{a} = \mathfrak{a}_G = \mathfrak{a}_P$  and that  $a = \text{id}_{\mathfrak{a}}$ . Moreover, it is natural to assume that  $\xi: \mathfrak{a}_G \rightarrow \mathfrak{g}$  is injective and so we may take  $\xi$  to be the inclusion. The image of  $\xi$  is an Abelian ideal, so it is reasonable to take  $\mathfrak{a} \subset \mathfrak{u}$  as  $f$ -invariant subspaces of  $\mathfrak{u}$  are such ideals.

The condition  $\omega_0 \neq 0$  is equivalent to the new algebra  $\mathfrak{h}$  obtained by the shear no longer being almost Abelian. Finally, for  $(\text{inc}, \text{id}|_{\mathfrak{a}}, \omega)$  to be shear data, we must have  $\xi^* \omega = 0$ , which is equivalent to  $\omega_0|_{\Lambda^2 \mathfrak{a}} = 0$ . This is ensured by the stronger condition  $\omega_0|_{\mathfrak{a} \otimes \mathfrak{u}} = 0$ , which is much simpler to work with.

*Proof of Proposition 4.2.* (a) By Remark 4.3, condition (iv) in Definition 3.6 is fulfilled since  $\omega_0|_{\mathfrak{u} \otimes \mathfrak{a}} = 0$ . Moreover, the discussion in §4.2 gives us that the validity of (iii) in Definition 3.6 for  $(\text{inc}, \text{id}_{\mathfrak{a}}, \omega)$  is equivalent to  $\gamma = \omega|_{\mathfrak{a} \otimes \mathfrak{g}} + \eta = -\alpha \otimes \nu|_{\mathfrak{a}} + \eta$ . As in §2.2, we see that  $\mathcal{L}_{\xi}^{\nabla} \beta = 0$  for all  $\beta \in \mathfrak{g}^*$  is equivalent to  $\gamma(X) = \xi \circ \gamma(X) = [X, \xi(\cdot)] = \text{ad}(X)|_{\mathfrak{a}}$  for all  $X \in \mathfrak{g}$ , and so to  $\gamma = \alpha \otimes f|_{\mathfrak{a}}$ . Then  $\eta = \alpha \otimes (f + \nu)|_{\mathfrak{a}}$ . The formulas for  $\eta$  and  $\gamma$  imply that the associated connections are flat and that  $(\text{inc}, \nabla)$  is torsion-free.

Finally, we have to check when  $d^{\nabla} \omega = 0$  holds. We have

$$0 = d^{\nabla} \omega = d\omega + \eta \wedge \omega = \alpha \wedge f.\omega_0 + \alpha \wedge (f + \nu) \circ \omega_0.$$

So  $d^{\nabla} \omega = 0$  if and only if  $f.\omega_0 = -(f + \nu) \circ \omega_0$ .

(b) By the formula for the differential  $d_{\mathfrak{h}} \psi$  in Corollary 3.11 and since  $d\psi = 0$ , we have to investigate when  $(\xi \lrcorner \psi) \wedge \omega$  is zero. First of all,

$$\begin{aligned} (\xi \lrcorner \psi) \wedge \omega &= (\xi \lrcorner \chi) \wedge \alpha \wedge \omega_0 + (\xi \lrcorner \tau) \wedge \alpha \wedge \nu + (\xi \lrcorner \tau) \wedge \omega_0 \\ &= ((\xi \lrcorner \chi) \wedge \omega_0 - (\xi \lrcorner \tau) \wedge \nu) \wedge \alpha + (\xi \lrcorner \tau) \wedge \omega_0 \\ &= (\kappa(\chi \wedge \omega_0) - (\xi \lrcorner \tau) \wedge \nu) \wedge \alpha + \kappa(\tau \wedge \omega_0), \end{aligned}$$

since  $\mathfrak{a}$  is in the kernel of  $\omega_0$ . Now also

$$\begin{aligned} -((\xi \lrcorner \tau) \wedge \nu)(X_1, \dots, X_r) &= -\sum_{i=1}^r (-1)^{r-i} \tau(\nu(X_i), X_1, \dots, \widehat{X_i}, \dots, X_r) \\ &= -(-1)^{r-1} \sum_{i=1}^r \tau(X_1, \dots, \nu(X_i), \dots, X_r) \\ &= (-1)^{r-1} (\nu.\tau)(X_1, \dots, X_r) \end{aligned}$$

for all  $X_1, \dots, X_r \in \mathfrak{g}$ , and the result follows.  $\square$

**4.3.1. Cocalibrated  $G_2$ -structures.** Suppose we have a  $G_2$ -structure  $\varphi \in \Lambda^3 \mathfrak{g}^*$  on a seven-dimensional almost Abelian Lie algebra  $\mathfrak{g}$  with Abelian ideal  $\mathfrak{u}$  of codimension 1. Choosing some unit-length  $\alpha \in \mathfrak{g}^*$  in the annihilator of  $\mathfrak{u}$ , it is well-known, cf. e.g. [MC06], that  $\varphi$  naturally induces a special almost Hermitian structure  $(\sigma, \rho) \in \Lambda^2 \mathfrak{u}^* \times \Lambda^3 \mathfrak{u}^*$  on  $\mathfrak{u}$  with

$$\star_{\varphi} \varphi = \frac{1}{2} \sigma^2 + \rho \wedge \alpha.$$

We use the convention that the two-form  $\sigma$ , the induced almost complex structure  $J$  and the induced Riemannian metric  $g$  are related by  $\sigma = g(\cdot, J\cdot)$ . Put  $X \in \mathfrak{g} \setminus \mathfrak{u}$  to be the unique element orthogonal to  $\mathfrak{u}$  with  $\alpha(X) = 1$ .

Suppose  $\varphi$  is cocalibrated  $d\star_{\varphi} \varphi = 0$ . Then  $f := \text{ad}(X)|_{\mathfrak{u}}$  lies in  $\mathfrak{sp}(\mathfrak{u}, \sigma)$  by [Fre12], so  $\text{tr}(f) = 0$ . Consider  $\omega = \omega_0 + \alpha \wedge \nu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{a}$  fulfilling the requirements of

Proposition 4.2 with respect to some  $\mathfrak{a} \subset \mathfrak{u}$ . We aim at a partial classification of all such  $\omega$  for which  $(\text{inc}, \text{id}_{\mathfrak{a}}, \omega)$  is left-invariant shear data with the shear  $\varphi_{\mathfrak{h}}$  of  $\varphi$  cocalibrated.

First note that the  $\mathfrak{gl}(\mathfrak{a})$ -valued left-invariant one forms  $\eta$  and  $\gamma$  which define the flat connections are fixed by equation (4.2). Furthermore, as  $\mathfrak{u}$  is six-dimensional,  $\kappa: \Lambda^6 \mathfrak{u}^* \otimes \mathfrak{a} \rightarrow \Lambda^5 \mathfrak{u}^*$  is injective. It follows that the first equation in (4.3) is given by  $\sigma^2 \wedge \omega_0 = 0$ , which in turn is equivalent to

$$\omega_0 \in \sigma^\perp \otimes \mathfrak{a} = [\Lambda_0^{1,1} \mathfrak{u}^*] \otimes \mathfrak{a} + [\Lambda^{2,0} \mathfrak{u}^*] \otimes \mathfrak{a}.$$

So by Proposition 4.2 we are left with solving the second equation in (4.3) and the first one in (4.2). This is complicated, so we restrict to certain special cases and obtain the following result.

**Proposition 4.4.** *Let  $(\mathfrak{g}, \mathfrak{u}, \varphi)$  be a seven-dimensional almost Abelian Lie algebra with a cocalibrated  $G_2$ -structure  $\varphi \in \Lambda^3 \mathfrak{g}^*$ . Let  $\mathfrak{a}$  be an  $f$ -invariant subspace of  $\mathfrak{u}$  and  $\omega = \omega_0 + \alpha \wedge \nu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{a}$  be as in Proposition 4.2.*

- (a) *If  $\omega_0 \in [\Lambda_0^{1,1} \mathfrak{u}^*] \otimes \mathfrak{a}$ , then  $(\text{inc}, \text{id}_{\mathfrak{a}}, \omega)$  shears  $(\mathfrak{g}, \varphi)$  to a cocalibrated structure if and only if  $\dim(\mathfrak{a}) \leq 2$ ,  $\nu \in \mathfrak{sp}(\mathfrak{u}, \sigma)$  and either*
  - (i)  *$\dim(\text{im}(\omega_0)) = 1$ ,  $f.\omega_0 = 0$  and  $\nu|_{\text{im}(\omega_0)} = -f|_{\text{im}(\omega_0)}$ , or*
  - (ii)  *$\dim(\text{im}(\omega_0)) = 2$ ,  $\omega_0 = \sum_{i=1}^2 \tilde{\omega}_i \otimes Y_i$ , with  $Y_1, Y_2 \in \mathfrak{a}$  a basis,  $\tilde{\omega}_i \wedge \tilde{\omega}_j = \delta_{ij} \tilde{\omega}_1^2$  for  $i, j \in \{1, 2\}$ ,  $f.\tilde{\omega}_1 = a \tilde{\omega}_2$ ,  $f.\tilde{\omega}_2 = -a \tilde{\omega}_1$ ,  $\nu(Y_1) = a Y_2 - f(Y_1)$  and  $\nu(Y_2) = -a Y_1 - f(Y_2)$  for some  $a \in \mathbb{R}$ .*
- (b) *If  $\dim(\mathfrak{a}) = 4$  and  $J(\text{im}(\omega_0)) \perp \mathfrak{a}$ , then  $(\text{inc}, \text{id}_{\mathfrak{a}}, \omega)$  shears  $(\mathfrak{g}, \varphi)$  to a cocalibrated structure if and only if  $f|_{\text{im}(\omega_0)} = -\text{tr}(f|_{\mathfrak{a}}) \text{id}_{\text{im}(\omega_0)}$  and*

$$\nu \in \tilde{\nu} + \{ \hat{\nu} \in \mathfrak{sp}(\mathfrak{u}, \sigma) \mid \hat{\nu}(\mathfrak{u}) \subset \mathfrak{a}, \hat{\nu}|_{\text{im}(\omega_0)} = 0 \},$$

where  $\tilde{\nu} \in \text{End}(\mathfrak{u})$  is  $\tilde{\nu}(W) = -\rho(JW, \kappa(J \circ \omega_0)^\sharp, \cdot)^\sharp$  for  $W \in J \text{im} \omega_0$  and  $\tilde{\nu}|_{(J \text{im} \omega_0)^\perp} = 0$ .

- (c) *If  $\dim(\mathfrak{a}) = 4$  and  $J(\text{im}(\omega_0)) \subset \mathfrak{a}$ , then  $(\text{inc}, \text{id}_{\mathfrak{a}}, \omega)$  shears  $(\mathfrak{g}, \varphi)$  to a cocalibrated structure if and only if  $\mathfrak{a}$  is a  $\sigma$ -degenerate subspace of  $\mathfrak{u}$  and*

$$\nu \in \tilde{\nu} + \{ \hat{\nu} \in \mathfrak{sp}(\mathfrak{u}, \sigma) \mid \hat{\nu}(\mathfrak{u}) \subset \mathfrak{a}, \hat{\nu}|_{\text{im}(\omega_0)} = 0 \},$$

where  $\tilde{\nu}$  is given for  $Y \in \text{im}(\omega_0)$  with  $\|Y\| = 1$  by

$$\tilde{\nu}(Y) = -\text{tr}(f|_{\mathfrak{a}})Y - f(Y),$$

$$\tilde{\nu}(JY) = (-\sigma(f(Y), JY) + \text{tr}(f|_{\mathfrak{a}}) - \mu) JY,$$

$$\tilde{\nu}|_{\mathfrak{a}^\perp} = -\sigma(f(Y), \cdot)|_{\mathfrak{a}^\perp} JY,$$

and  $\tilde{\nu}|_U = \mu \text{id}_U$  on  $U = (\text{im} \omega_0 + J \text{im} \omega_0)^\perp \cap \mathfrak{a}$ , with  $\mu \in \mathbb{R}$  fixed by  $\kappa(\omega_0 \wedge \rho)|_{\Lambda^4 \text{span}(Y, JY)^\perp} = -\mu \sigma^2|_{\Lambda^4 \text{span}(Y, JY)^\perp}$ .

*Proof.* (a) Here,  $\rho \wedge \omega_0 = 0$  and so the second equation in (4.3) simplifies to  $\nu \cdot \sigma \wedge \sigma = 0$ , i.e. to  $\nu \cdot \sigma = 0$ , since the Lefschetz operator is bijective on two-forms in six dimensions. Thus we regard  $\nu$  as an endomorphism of  $\mathfrak{u}$  and see that  $\nu \in \mathfrak{sp}(\mathfrak{u}, \sigma)$ .

Let  $\star_{\mathfrak{u}}$  be the Hodge star operator on  $\mathfrak{u}$ . Then using Schur's Lemma and a concrete element, we find  $\star_{\mathfrak{u}} \tilde{\omega} = -\tilde{\omega} \wedge \sigma$  for each  $\tilde{\omega} \in [\Lambda_0^{1,1}]$ . Hence, for such  $\tilde{\omega}$ ,  $\sigma \wedge \tilde{\omega}^2 = -\tilde{\omega} \wedge \star_{\mathfrak{u}} \tilde{\omega} = -g(\tilde{\omega}, \tilde{\omega}) \frac{1}{6} \sigma^3$ , showing that  $\sigma \wedge \tilde{\omega}^2$  is non-zero if  $\tilde{\omega}$  is non-zero. In particular, any non-zero element in  $[\Lambda_0^{1,1}]$  has rank at least four. The condition  $\omega_0|_{\mathfrak{a} \otimes \mathfrak{u}} = 0$ , then gives  $\dim(\mathfrak{a}) \leq 2$ , and so  $\dim(\text{im}(\omega_0)) \leq 2$ .

If  $\dim(\text{im}(\omega_0)) = 1$ , then  $(f + \nu) \circ \omega_0 = \lambda \omega_0$  for some  $\lambda \in \mathbb{R}$ , and the first equation in (4.2) gives  $f.\omega_0 = -\lambda \omega_0$ . Now  $0 \neq \sigma \wedge \omega_0^2 \in \Lambda^6 \mathfrak{u}^* \otimes \mathfrak{a}^{\otimes 2}$  and so, recalling that  $\text{tr}(f) = 0$ , we have

$$0 = \text{tr}(f) \sigma \wedge \omega_0^2 = f.(\sigma \wedge \omega_0^2) = 2\sigma \wedge f.\omega_0 \wedge \omega_0 = -2\lambda \sigma \wedge \omega_0^2,$$

giving  $\lambda = 0$ . Thus,  $f.\omega_0 = 0$  and  $\nu|_{\text{im}(\omega_0)} = -f|_{\text{im}(\omega_0)}$ .

Let us now consider the case  $\dim(\text{im}(\omega_0)) = 2$ . Then  $\dim(\mathfrak{a}) = 2$  and  $\mathfrak{a}$  is a  $J$ -invariant subspace as it is the kernel of a  $(1,1)$ -form. It follows that  $\text{tr}(f|_{\mathfrak{a}}) = 0$ . As the space of four-forms on  $\mathfrak{u}$  with annihilator  $\mathfrak{a}$  is one-dimensional and the square of any non-zero element in  $[\Lambda_0^{1,1}\mathfrak{u}^*]$  which annihilates  $\mathfrak{a}$  is non-zero, we may choose a basis  $Y_1, Y_2$  of  $\mathfrak{a}$ , so that  $\omega_0 = \tilde{\omega}_1 \otimes Y_1 + \tilde{\omega}_2 \otimes Y_2$  with  $\tilde{\omega}_1, \tilde{\omega}_2 \in [\Lambda_0^{1,1}\mathfrak{u}^*]$  annihilating  $\mathfrak{a}$  and satisfying  $\tilde{\omega}_i \wedge \tilde{\omega}_j = \delta_{ij} \tilde{\omega}_1^2 \neq 0$ . Since  $\text{tr}(f|_{\mathfrak{a}}) = 0$  and  $\text{tr}(f) = 0$ , we have  $f \cdot (\tilde{\omega}_i \wedge \tilde{\omega}_j) = 0$  for all  $i, j = 1, 2$ . Moreover, the first equation in (4.2) yields  $f \cdot \tilde{\omega}_i \in \text{span}(\tilde{\omega}_1, \tilde{\omega}_2)$  for  $i = 1, 2$ . Hence, we obtain  $f \cdot \tilde{\omega}_1 = a \tilde{\omega}_2$  and  $f \cdot \tilde{\omega}_2 = -a \tilde{\omega}_1$  for some  $a \in \mathbb{R}$ . But then the first equation in (4.2) is equivalent to  $\nu(Y_1) = aY_2 - f(Y_1)$  and  $\nu(Y_2) = -aY_1 - f(Y_2)$ .

(b) & (c) For these cases, note that  $\omega_0$  has kernel equal to  $\mathfrak{a}$ , that  $\omega_0^2 = 0$  and that  $\dim(\text{im}(\omega_0)) = 1$ . So the equation  $\sigma^2 \wedge \omega_0 = 0$  is equivalent to  $\mathfrak{a}$  being a  $\sigma$ -degenerate subspace, as claimed.

The  $f$ -invariance of  $\mathfrak{a}$  and  $\text{tr}(f) = 0$  give  $f \cdot \omega_0 = \text{tr}(f|_{\mathfrak{a}}) \omega_0$ . So the first equation in (4.2) is equivalent to  $\nu|_{\text{im}(\omega_0)} = -\text{tr}(f|_{\mathfrak{a}}) \text{id}_{\text{im}(\omega_0)} - f|_{\text{im}(\omega_0)}$  and we are left with solving the equation

$$\kappa(\omega_0 \wedge \rho) = \nu \cdot \sigma \wedge \sigma. \quad (4.4)$$

Note that the space of  $\nu: \mathfrak{u} \rightarrow \mathfrak{a} \subset \mathfrak{u}$  solving equation (4.4) and with  $\nu|_{\text{im}(\omega_0)}$  given as above is an affine subspace of  $\text{End}(\mathfrak{u})$  modelled on  $\{\hat{\nu} \in \mathfrak{sp}(\mathfrak{u}, \sigma) \mid \hat{\nu}(\mathfrak{u}) \subset \mathfrak{a}, \hat{\nu}|_{\text{im}(\omega_0)} = 0\}$ .

(b) Before checking that  $\tilde{\nu}$  as in the statement is a solution of equation (4.4), we show that any solution  $\nu$  of equation (4.4) has to fulfil  $\nu|_{\text{im}(\omega_0)} = 0$  and so we must have  $f|_{\text{im}(\omega_0)} = -\text{tr}(f|_{\mathfrak{a}}) \text{id}_{\text{im}(\omega_0)}$ . As  $\dim(\text{im}(\omega_0)) = 1$ ,  $\text{im}(\omega_0)$  lies in the kernel of  $\kappa(\omega_0 \wedge \rho)$ . Let  $Y$  be a non-zero element of  $\text{im}(\omega_0)$ . Then  $\omega_0 = \tilde{\omega} \otimes Y$ ,  $\tilde{\omega}$  has kernel  $\mathfrak{a}$  and  $\kappa(\omega_0 \wedge \rho) = \tilde{\omega} \wedge (Y \lrcorner \rho)$ . Hence,  $0 = Y \lrcorner \kappa(\omega_0 \wedge \rho) = (Y \lrcorner \nu \cdot \sigma) \wedge \sigma + \nu \cdot \sigma \wedge (Y \lrcorner \sigma)$ . We define

$$Z = \text{span}(Y, JY)^\perp \subset \mathfrak{u}.$$

Restricting the previous identity to  $\Lambda^3 Z$ , we get  $Y \lrcorner \nu \cdot \sigma|_Z = 0$ . As  $\text{im}(\nu) \subset \mathfrak{a} \subset (JY)^\perp$ , this implies  $\nu(Y) \in \text{span}(Y, JY) \cap \mathfrak{a} = \text{span}(Y)$ . So  $\nu(Y) = \lambda Y$  for some  $\lambda \in \mathbb{R}$ . Then  $0 = (\nu \cdot \sigma \wedge \sigma)(Y, JY, \cdot, \cdot)|_{\Lambda^2 Z}$  gives us  $\nu \cdot \sigma = \lambda \sigma$  on  $\Lambda^2 Z$ . Now the kernel of the four-form  $\nu \cdot \sigma \wedge \sigma$  has to be at least two-dimensional. Moreover,  $(\nu \cdot \sigma \wedge \sigma)|_{\Lambda^2 \mathfrak{a}^\perp \wedge \Lambda^2 \mathfrak{a}} = \tilde{\omega}|_{\Lambda^2 \mathfrak{a}^\perp} \wedge (Y \lrcorner \rho)|_{\Lambda^2 \mathfrak{a}} \neq 0$  and  $(\nu \cdot \sigma \wedge \sigma)|_{\mathfrak{a}^\perp \wedge \Lambda^3 \mathfrak{a}} = 0$ . So the kernel of  $\nu \cdot \sigma \wedge \sigma$  is contained in  $\mathfrak{a}$  and, hence, has non-zero intersection with  $Z$ . Thus,  $0 = (\nu \cdot \sigma \wedge \sigma)|_{\Lambda^4 Z} = \lambda \sigma^2|_{\Lambda^4 Z}$  giving  $\lambda = 0$  as claimed.

Now consider  $\tilde{\nu}$ . Let  $W \in J \text{im}(\omega_0)$  be non-zero. We first note that  $\tilde{\nu}(W) \in \mathfrak{a}$ , as  $\mathfrak{a}$  is in the kernel of  $\omega_0$ , so  $\kappa(J \circ \omega_0)^\sharp \in (\mathfrak{a} \oplus J \text{im}(\omega_0))^\perp$ , which gives  $\tilde{\nu}(W) = -\rho(JW, \kappa(J \circ \omega_0)^\sharp, \cdot)^\sharp \in \text{span}(W, \kappa(J \circ \omega_0)^\sharp)^\perp = \mathfrak{a}$ . Moreover, both  $\kappa(\omega_0 \wedge \rho)$  and  $\tilde{\nu} \cdot \sigma \wedge \sigma$  are zero when we restrict to  $\Lambda^4(J \text{im}(\omega_0))^\perp$ .

Besides,  $(\sigma \wedge Y \lrcorner \rho)|_{\Lambda^4 \text{span}(JY)^\perp} = 0$ . Since  $J(JY \lrcorner \tilde{\omega})^\sharp = J\kappa(J \circ \omega_0)^\sharp \in \text{span}(JY)^\perp$ , we get on  $\Lambda^3(J \text{im}(\omega_0))^\perp = \Lambda^3 \text{span}(JY)^\perp$  that

$$\begin{aligned} JY \lrcorner (\tilde{\nu} \cdot \sigma \wedge \sigma) &= -\sigma(\tilde{\nu}(JY), \cdot) \wedge \sigma = -g(\rho(Y, \kappa(J \circ \omega_0)^\sharp, \cdot)^\sharp, J \cdot) \wedge \sigma \\ &= -\sigma \wedge \rho(Y, J(JY \lrcorner \tilde{\omega})^\sharp, \cdot) = \sigma(J(JY \lrcorner \tilde{\omega})^\sharp, \cdot) \wedge (Y \lrcorner \rho) \\ &= g((JY \lrcorner \tilde{\omega})^\sharp, \cdot) \wedge (Y \lrcorner \rho) = (JY \lrcorner \tilde{\omega}) \wedge (Y \lrcorner \rho) = JY \lrcorner \kappa(\omega_0 \wedge \rho), \end{aligned}$$

so (4.4) is satisfied.

(c) Using the results from above, we only have to show that  $\tilde{\nu}$  defined as in Proposition 4.4(c) solves equation (4.4). Now, as  $JY \in \mathfrak{a}$ , the left-hand side is zero if we insert  $Y$  or  $JY$ . Since  $\mathfrak{a}^\perp = JU$ , straightforward computations show

$$\begin{aligned} \tilde{\nu} \cdot \sigma(Y, JY) &= \mu \sigma(Y, JY), & \tilde{\nu} \cdot \sigma|_{\Lambda^2 U} &= 0 = \tilde{\nu} \cdot \sigma|_{\Lambda^2 JU}, \\ \tilde{\nu} \cdot \sigma|_{\text{span}(Y, JY) \wedge (U \oplus JU)} &= 0, & \tilde{\nu} \cdot \sigma|_{U \wedge JU} &= -\mu \sigma|_{U \wedge JU}. \end{aligned}$$

So  $\tilde{\nu} \cdot \sigma \wedge \sigma$  is also zero if we insert  $Y$  or  $JY$ . Finally, on  $\Lambda^4(U \oplus JU)$  we have  $\tilde{\nu} \cdot \sigma \wedge \sigma = -\mu \sigma^2 = \kappa(\omega_0 \wedge \rho)$  as required, since  $U \oplus JU = \text{span}(Y, JY)^\perp$ .  $\square$

Let us give examples of all cases in Proposition 4.4.

**Example 4.5.** Look at the almost Abelian Lie algebra defined by

$$(a_1.17, a_2.27, a_3.37, -a_1.47, -a_2.57, -a_3.67, 0)$$

for  $a_1, a_2, a_3 \in \mathbb{R}$ , where  $a_1.17$  in place 1 means that  $de^1 = a_1 e^{17}$  with respect to the basis  $e^1, \dots, e^7$  of  $\mathfrak{g}^*$ , etc. Consider the cocalibrated  $G_2$ -structure  $\varphi \in \Lambda^3 \mathfrak{g}^*$  with closed Hodge dual  $\star_\varphi \varphi = 1425 + 1436 + 2536 + 1237 - 1567 + 2467 - 3457$ , where  $1425 := e^{1425} := e^1 \wedge e^4 \wedge e^2 \wedge e^5$ , etc.

Case (a)(i): Taking  $\mathfrak{a} = \text{span}(e_1, e_4)$ ,  $\omega_0 = (e^{36} - e^{25}) \otimes e_1$  and  $\nu(e_1) = -a_1 e_1$ ,  $\nu(e_4) = a_1 e_4$  and  $\nu(e_i) = 0$  for  $i \in \{2, 3, 5, 6\}$ , the shear gives a cocalibrated  $G_2$ -structure on

$$(25 - 36, a_2.27, a_3.37, 0, -a_2.57, -a_3.67, 0).$$

Case (b): Assume that  $a_3 = -2a_1$ . Then, taking  $\mathfrak{a} = \text{span}(e_1, e_2, e_3, e_5)$ ,  $\omega_0 = -e^{46} \otimes e_1$  and  $\nu(e_4) = -e_5$ ,  $\nu(e_i) = 0$  for all  $i \in \{1, 2, 3, 5, 6\}$ , the shear gives a cocalibrated  $G_2$ -structure on

$$(a_1.17 + 46, a_2.27, -2a_1.37, -a_1.47, -47 - a_2.57, 2a_1.67, 0).$$

Case (c): Taking  $\mathfrak{a} = \text{span}(e_1, e_4, e_5, e_6)$ ,  $\omega_0 = -ce^{23} \otimes e_1$  for some  $c \in \mathbb{R}$  and  $\nu(e_1) = (a_2 + a_3 - a_1)e_1$ ,  $\nu(e_4) = (a_1 - a_2 - a_3 - \frac{c}{2})e_4$ ,  $\nu(e_i) = \frac{c}{2}e_i$  for  $i = 5, 6$  and  $\nu(e_j) = 0$  for  $j = 2, 3$  and setting  $b = a_2 + a_3$ , the shear gives a cocalibrated  $G_2$ -structure on

$$\left(b.17 + c.23, a_2.27, a_3.37, -(b + \frac{c}{2}).47, (\frac{c}{2} - a_2).57, (\frac{c}{2} - a_3).67, 0\right).$$

For Case (a)(ii) we need to start with a different Lie algebra. We take

$$(a.47, -a.57, b.37, -a.17, a.27, -b.67, 0)$$

and the cocalibrated  $G_2$ -structure  $\varphi \in \Lambda^3 \mathfrak{g}^*$  given by the same formula as above. Moreover, let  $\mathfrak{a} = \text{span}(e_3, e_6)$ ,  $\omega_0 = -(e^{12} + e^{45}) \otimes e_3 - (e^{15} + e^{24}) \otimes e_6$  and  $\nu \in \mathfrak{g}^* \otimes \mathfrak{a}$  defined by  $\nu(e_3) = -be_3 + 2ae_6$ ,  $\nu(e_6) = -2ae_3 + be_6$ . Then  $f.(e^{12} + e^{45}) = 2a(e^{15} + e^{24})$  and  $f.(e^{15} + e^{24}) = -2a(e^{12} + e^{45})$ , so the shear gives a cocalibrated  $G_2$ -structure on

$$(a.47, -a.57, -2a.67 + 12 + 45, -a.17, a.27, 2a.37 + 15 + 24, 0).$$

**4.3.2. Calibrated  $G_2$ -structures.** Given a  $G_2$ -structure  $\varphi$  on an almost Abelian Lie algebra  $\mathfrak{g}$  and unit-length  $\alpha \in \mathfrak{g}^*$  in the annihilator of  $\mathfrak{u}$ , there is also an almost Hermitian structure  $(\sigma, \rho)$  on the codimension one Abelian ideal  $\mathfrak{u}$  related to  $\varphi$  via  $\varphi = \sigma \wedge \alpha + \rho$ , cf. [MC06]. The *calibrated* case is when  $d\varphi = 0$ . In this situation  $f = \text{ad}(X)|_{\mathfrak{u}} \in \mathfrak{sl}(\mathfrak{u}, \rho) := \{g \in \text{End}(\mathfrak{u}) \mid g \cdot \rho = 0\} \cong \mathfrak{sl}(3, \mathbb{C})$  and so  $\text{tr}(f) = 0$  and  $[f, J] = 0$ , cf. [Fre13]. We aim at a partial classification of left-invariant shear data  $(\text{inc}, \text{id}_{\mathfrak{a}}, \omega)$  as in Proposition 4.2 for which the shear of  $\varphi$  is again calibrated.

When  $U$  is a subspace of  $\mathfrak{u}$ , we write  $\text{proj}_U$  for the orthogonal projection  $\mathfrak{u} \rightarrow U$ .

**Proposition 4.6.** *Let  $(\mathfrak{g}, \mathfrak{u}, \varphi)$  be a seven-dimensional almost Abelian Lie algebra with a calibrated  $G_2$ -structure. Fix an  $f$ -invariant subspace  $\mathfrak{a}$  of  $\mathfrak{u}$  and let  $\omega = \omega_0 + \alpha \wedge \nu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{a}$  be as in Proposition 4.2.*

(a) *If  $\ker(\omega_0)$  is  $J$ -invariant and of dimension 2, then  $(\text{inc}, \text{id}_{\mathfrak{a}}, \omega)$  shears  $(\mathfrak{g}, \varphi)$  to a calibrated structure if and only if  $\dim(\mathfrak{a}) = 2$  and for some  $Y \in \mathfrak{a} \setminus \{0\}$  either*

(i)  $\omega_0 = \text{tr}(f|_{\mathfrak{a}})(JY \lrcorner \rho \otimes Y - Y \lrcorner \rho \otimes JY)$  and

$$\nu \in -\text{tr}(f|_{\mathfrak{a}}) \text{proj}_{\mathfrak{a}} + \{\hat{\nu} \in \mathfrak{sl}(\mathfrak{u}, \rho) \mid \hat{\nu}|_{\mathfrak{a}} = 0, \hat{\nu}(\mathfrak{u}) \subset \mathfrak{a}\},$$

or

(ii)  $f|_{\mathfrak{a}} = 0$ ,  $\omega_0 = (aY \lrcorner \rho + bJY \lrcorner \rho) \otimes Y + (cY \lrcorner \rho - aJY \lrcorner \rho) \otimes JY$  for  $(a, b, c) \in \mathbb{R}^3 \setminus \{0\}$  with  $a^2 + bc = 0$  and

$$\nu \in \tilde{\nu} + \{ \hat{\nu} \in \mathfrak{sl}(\mathfrak{u}, \rho) \mid \hat{\nu}|_{\mathfrak{a}} = 0, \hat{\nu}(\mathfrak{u}) \subset \mathfrak{a} \}$$

with  $\tilde{\nu}(Y) = cY - aJY$ ,  $\tilde{\nu}(JY) = -aY - bJY$ ,  $\tilde{\nu}(\mathfrak{a}^\perp) = \{0\}$ .

(b) If  $\dim(\mathfrak{a}) = 4$  and  $\mathfrak{a} = \ker(\omega_0)$  is  $J$ -invariant, then  $(\text{inc}, \text{id}_{\mathfrak{a}}, \omega)$  shears  $(\mathfrak{g}, \varphi)$  to a calibrated structure if and only if  $\omega_0 = \tilde{\omega} \otimes Y$  for a unit length  $Y \in \mathfrak{a}$  with  $\|Y\| = 1$  and  $\tilde{\omega} \in \Lambda^2 \mathfrak{u}^*$  decomposable, and

$$\nu = \tilde{\nu} + \{ \hat{\nu} \in \mathfrak{sl}(\mathfrak{u}, \rho) \mid \hat{\nu}(Y) = 0, \hat{\nu}(\mathfrak{u}) \subset \mathfrak{a} \}$$

where  $\tilde{\nu}$  is given by

$$\tilde{\nu}(Y) = -\text{tr}(f|_{\mathfrak{a}})Y - f(Y), \quad \tilde{\nu}(JY) = \mu Y - \lambda JY - J\text{proj}_U(f(Y)),$$

$$\tilde{\nu}|_U = \lambda \text{id}_U + \mu J|_U, \quad \tilde{\nu}(W) = -\frac{\tilde{\omega}(W, JW)}{2\|\rho(Y, W, \cdot)\|^2} \rho(Y, W, \cdot)^\# \quad \forall W \in \mathfrak{a}^\perp$$

with  $U = \text{span}(Y, JY)^\perp \cap \mathfrak{a}$ ,  $\lambda = \text{tr}(f|_{\mathfrak{a}}) + g(f(Y), Y)$  and  $\mu = g(f(Y), JY)$ .

(c) If  $\dim(\mathfrak{a}) = 4$  and  $\mathfrak{a} = \ker(\omega_0)$  is not  $J$ -invariant, then  $(\text{inc}, \text{id}_{\mathfrak{a}}, \omega)$  shears  $(\mathfrak{g}, \varphi)$  to a calibrated structure if and only if  $\omega_0 = \tilde{\omega} \otimes Y$  for a unit length  $Y \in \mathfrak{a} \cap J\mathfrak{a}$  with  $Y \lrcorner \rho|_{\Lambda^2 \mathfrak{a}} = 0$  and decomposable  $\tilde{\omega} \in \mu JY \lrcorner \rho + \llbracket \Lambda^{1,1} \mathfrak{u}^* \rrbracket$ , such that  $\mu \in \mathbb{R}$  is non-zero,  $f(Y) = \lambda Y$  for  $\lambda \in \mathbb{R}$  fulfilling  $4\lambda JY \lrcorner \rho|_{\Lambda^2 U} = -J^* \tilde{\omega}|_{\Lambda^2 U}$  and

$$\nu = \tilde{\nu} + \{ \hat{\nu} \in \text{End}(\mathfrak{u}) \mid [\hat{\nu}, J] = 0, \hat{\nu}|_{\text{span}(Y, JY)} = 0, \hat{\nu}(\mathfrak{u}) \subset \text{span}(Y, JY) \}$$

for  $\tilde{\nu} = \text{diag}(-2\lambda, -4\lambda, 2\lambda, 0)$  with respect to  $\mathfrak{u} = \text{span}(Y) \oplus \text{span}(JY) \oplus U \oplus JU$ ,  $U = \text{span}(Y, JY)^\perp \cap \mathfrak{a}$ .

*Proof.* (a) First observe that we always have  $\text{im } \omega_0 \subset \mathfrak{a} \subset \ker \omega_0$ . Consider the case  $\dim(\mathfrak{a}) = 2$ . Choose  $Y_1 \in \ker(\omega_0) = \mathfrak{a}$  and set  $Y_2 = JY_1$ . We may then write  $\omega_0 = \sum_{i=1}^2 \omega_i \otimes Y_i$  for two-forms  $\omega_1, \omega_2$ , where  $\omega_2 = 0$  in the case  $\dim(\text{im } \omega_0) = 1$ . The second equation in (4.3) reads

$$Y_2^b \wedge \omega_1 - Y_1^b \wedge \omega_2 = -\kappa(\sigma \wedge \omega_0) = \nu \cdot \rho. \quad (4.5)$$

As  $\nu(\mathfrak{u}) \subset \mathfrak{a}$  and  $\rho(Z, JZ, \cdot) = 0$  for all  $Z \in \mathfrak{u}$ , inserting  $Y_{3-i}$  into equation (4.5) gives  $\omega_i \in \{\tilde{\omega}_1, \tilde{\omega}_2\}$  for  $\tilde{\omega}_i := Y_i \lrcorner \rho$ ,  $i = 1, 2$ . Hence, we may write  $\omega_i = \sum_{j=1}^2 a_{ij} \tilde{\omega}_j$  for some  $A = (a_{ij}) \in M_2(\mathbb{R})$ . Equation (4.5) then gives  $\nu(Y_j) = \sum_{i=1}^2 c_{ij} Y_i$  with  $C = (c_{ij}) = A^T D$  for  $D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ .

Since  $\tilde{\omega}_i \wedge \tilde{\omega}_j = \delta_{ij} \tilde{\omega}_1^2$  for all  $i, j = 1, 2$ , the first equation in (4.3) reads

$$0 = \kappa(\rho \wedge \omega_0) = \sum_{i=1}^2 \tilde{\omega}_i \wedge \omega_i = (a_{11} + a_{22}) \tilde{\omega}_1^2. \quad (4.6)$$

So  $\text{tr } A = 0$ , giving  $A \in \mathfrak{sl}(2, \mathbb{R})$ .

If  $\dim(\mathfrak{a}) = 1$ , then  $\omega_2 = 0$  and  $\omega_1 = a_{11} \tilde{\omega}_1$ . But (4.6) gives  $a_{11} = 0$  and so  $\omega_0 = 0$ , which is a contradiction. Thus  $\dim(\mathfrak{a}) = 2$ .

As  $\mathfrak{a}$  is  $f$ -invariant and  $[f, J] = 0$ , we have  $f(Y_j) = \sum_{i=1}^2 b_{ij} Y_i$  with  $B = (b_{ij}) = \begin{pmatrix} b_{11} & b_{12} \\ -b_{12} & b_{11} \end{pmatrix} \in M_1(\mathbb{C}) \subset M_2(\mathbb{R})$ , i.e.  $b_{21} = -b_{12}$  and  $b_{22} = b_{11}$ . Since  $f \cdot \rho = 0$ , we also get  $f \cdot \tilde{\omega}_j = \sum_{i=1}^2 b_{ij} \tilde{\omega}_i$ . Now the first equation in (4.2) is equivalent to  $AB^T = -(B+C)A = -BA - A^T D A = -BA - \det(A)D$ , which is

$$2a_{11}b_{11} = -(a_{12} + a_{21})b_{12}, \quad 2a_{11}b_{12} = (a_{12} + a_{21})b_{11}, \\ \det(A) = 2(a_{12}b_{11} - a_{11}b_{12}).$$

The first two equations are solved if and only if  $a_{12} + a_{21} = 0 = a_{11}$  or  $B = 0$ . In the first case, we must have  $a_{12} \neq 0$ , as  $A \neq 0$ , and the last equation implies  $a_{12} = 2b_{11} = \text{tr}(f|_{\mathfrak{a}})$ . In the second case, we have  $f|_{\mathfrak{a}} = 0$  and the last equation gives us  $a_{11}^2 + a_{12}a_{21} = -\det(A) = 0$ . This gives the two cases claimed.

(b) & (c) First note that  $\dim \ker \omega_0 = \dim \mathfrak{a} = 4$ , implies that  $\dim \text{im } \omega_0 \leq \dim \Lambda^2(\ker \omega_0^\perp) = 1$ . We thus have  $\omega_0 = \tilde{\omega} \otimes Y$  for some non-zero decomposable two-form  $\tilde{\omega}$  and a  $Y \in \mathfrak{a}$  with  $\|Y\| = 1$ . Now the first equation in (4.3) reads  $Y \lrcorner \rho \wedge \tilde{\omega} = 0$  and the second equation in (4.3) reads  $JY^b \wedge \tilde{\omega} = \nu \cdot \rho$ . Moreover, using  $f \cdot \omega_0 = \text{tr}(f|_{\mathfrak{a}})\omega_0$  we see that the first equation in (4.2) is equivalent to  $\nu(Y) = -f(Y) - \text{tr}(f|_{\mathfrak{a}})Y$ .

(b) Here,  $Y \lrcorner \rho \wedge \tilde{\omega} = Y \lrcorner (\rho \wedge \tilde{\omega}) = 0$  holds as  $\mathfrak{a} = \ker(\tilde{\omega})$  is  $J$ -invariant, i.e.  $\tilde{\omega}$  is a  $(1,1)$ -form. So we only have to check that the given  $\tilde{\nu}$  is a solution of  $JY^b \wedge \tilde{\omega} = \tilde{\nu} \cdot \rho$ . To simplify the notation, we set

$$a(W) = \frac{\tilde{\omega}(W, JW)}{2\|\rho(Y, W, \cdot)\|^2} = \frac{\tilde{\omega}(W, JW)}{2\|\rho(JY, JW, \cdot)\|^2}$$

for  $W \in \mathfrak{a}^\perp$ . For such  $W$ , we have  $JW \in \mathfrak{a}^\perp$  too, so  $\tilde{\nu}(W) \in U \subset \mathfrak{u}$ . Moreover, as  $\rho$  is zero when evaluated on any pair  $A, JA$  and as  $f \cdot \rho = 0$ , straightforward computations give us that  $\tilde{\nu} \cdot \rho$  is zero on  $\Lambda^3 \mathfrak{a} + \Lambda^2 \mathfrak{a} \wedge \mathfrak{a}^\perp + U \wedge \Lambda^2 \mathfrak{a}^\perp$ . As  $\tilde{\nu}(JW) = -J\tilde{\nu}(W)$  for  $W \in \mathfrak{a}^\perp$ , we obtain  $(\tilde{\nu} \cdot \rho)(Z, W, JW) = 2\rho(Z, JW, \tilde{\nu}(W))$ . Thus

$$(\tilde{\nu} \cdot \rho)(Z, W, JW) = 2\rho(Z, JW, \tilde{\nu}(W)) = 2a(W)g(\rho(Z, JW, \cdot), \rho(JY, JW, \cdot))$$

for all  $W \in \mathfrak{a}^\perp$  and all  $Z \in \text{span}(Y, JY)$ . For  $Z = Y$ , this equals zero since  $\rho(Y, JW, \cdot) = J\rho(JY, JW, \cdot)$ ; for  $Z = JY$ , we obtain

$$(\tilde{\nu} \cdot \rho)(JY, W, JW) = \tilde{\omega}(W, JW) = (JY^b \wedge \tilde{\omega})(JY, W, JW),$$

as required.

(c) Inserting  $JY$  into  $Y \lrcorner \rho \wedge \tilde{\omega} = 0$ , we get  $JY \lrcorner \tilde{\omega} = 0$ , as  $Y \lrcorner \rho$  has rank four. So  $JY \in \ker(\omega_0) = \mathfrak{a}$  and the equation may be considered as one on the four-dimensional space  $\text{span}(Y, JY)^\perp$ . But then  $\tilde{\omega}$  solves  $Y \lrcorner \rho \wedge \tilde{\omega} = 0$  if and only if  $\tilde{\omega} \in [\Lambda^{1,1}(\text{span}(Y, JY)^\perp)^*] \oplus \text{span}(JY \lrcorner \rho)$ . As  $\mathfrak{a} = \ker(\tilde{\omega})$  is not  $J$ -invariant, the  $\text{span}(JY \lrcorner \rho)$ -part of  $\tilde{\omega}$  must be non-zero as claimed.

Assume now that  $JY^b \wedge \tilde{\omega} = \nu \cdot \rho$  holds. Inserting both  $Y$  and  $JY$  into this equation one obtains  $\rho(f(Y), JY, \cdot) - \rho(Y, \nu(JY), \cdot) = 0$  and so  $\rho(Y, \nu(JY) + Jf(Y), \cdot) = 0$ . Thus  $\nu(JY) = -Jf(Y)$  up to terms in  $\text{span}(Y, JY)$ . However, the hypotheses give  $JU \cap \mathfrak{a} = \{0\}$ , so  $f(Y), \nu(JY) \in \text{span}(Y, JY)$ .

Inserting  $Z \in U$  and  $JZ \notin \mathfrak{a}$  in to the same equation, we get  $\nu(JZ) = J\nu(Z)$  up to terms in  $\text{span}(Z, JZ)$ . So there are  $\lambda_1, \lambda_2 \in \mathbb{R}$  such that  $\nu(Z) - \lambda_1 Z, \nu(JZ) - \lambda_2 Z \in \text{span}(Y, JY)$  for all  $Z \in U$ . Take now  $Z, W \in U$  such that  $Y, Z, W$  is a  $\mathbb{C}$ -basis of  $\mathfrak{u}$ . Then  $\rho(Y, Z, W) = 0$  from  $Y \lrcorner \rho \wedge \tilde{\omega} = 0$ . So we have  $Y \lrcorner \rho|_{\Lambda^2 \mathfrak{a}} = 0$  and must have  $\rho(JY, Z, W) \neq 0$ . Hence,

$$0 = (JY^b \wedge \tilde{\omega})(Y, Z, W) = (\nu \cdot \rho)(Y, Z, W) = -\rho(\nu(Y), Z, W)$$

implies  $\nu(Y) \in \text{span}(Y)$ . So  $f(Y) = \lambda Y$  for some  $\lambda \in \mathbb{R}$ . As

$$0 = (f \cdot \rho)(JY, Z, W) = -(\text{tr}(f|_{\mathfrak{a}}) - \lambda)\rho(JY, Z, W),$$

we must have  $\text{tr}(f|_{\mathfrak{a}}) = \lambda$ . Thus  $\nu(Y) = -2\lambda Y$ .

Similarly,  $(\nu \cdot \rho)(Y, Z, JW) = 0$  yields  $\lambda_1 = 2\lambda$  and  $\nu \cdot \rho(JY, Z, W) = 0$  implies that  $\nu(JY) = aY - 4\lambda JY$  for some  $a \in \mathbb{R}$ . The equality  $0 = (\nu \cdot \rho)(Y, JZ, JW)$  gives us  $\lambda_2 = 0$  and then the equality  $0 = (\nu \cdot \rho)(JY, Z, JW)$  gives us  $a = 0$ .

Finally, inserting  $JY, JZ, JW$  into  $JY^b \wedge \tilde{\omega} = \nu \cdot \rho$ , we get  $4\lambda\rho(JY, Z, W) = -\tilde{\omega}(JZ, JW)$ . This gives that the difference  $\hat{\nu}$  between two solutions of  $JY^b \wedge \tilde{\omega} = \nu \cdot \rho$  are those  $\hat{\nu} \in \text{End}(\mathfrak{u})$  with  $\hat{\nu}|_{\text{span}(Y, JY)} = 0$ ,  $\hat{\nu}(\mathfrak{u}) \subset \text{span}(Y, JY)$  and  $\hat{\nu} \in \mathfrak{sl}(\mathfrak{u}, \rho)$ , which here is equivalent to  $[\hat{\nu}, J] = 0$ , as claimed. Conversely, the above computations show that  $\tilde{\nu}$  as in the statement fulfils  $JY^b \wedge \tilde{\omega} = \tilde{\nu} \cdot \rho$ , completing the proof.  $\square$

Let us give explicit examples for all the cases in Proposition 4.6.



**Example 4.7.** Start with the almost Abelian Lie algebra defined by

$$(a.17, a.27, b.37, b.47, c.57, c.67, 0)$$

for  $a, b, c \in \mathbb{R}$  with  $a + b + c = 0$  and consider the calibrated  $G_2$ -structure  $\varphi = 127 + 347 + 567 + 135 - 146 - 236 - 245$ .

Case (a)(i): Taking  $\mathfrak{a} = \text{span}(e_1, e_2)$ ,  $\omega_0 = 2a((e^{36} + e^{45}) \otimes e_1 + (e^{35} - e^{46}) \otimes e_2)$  and  $\nu \in \text{End}(\mathfrak{u})$  defined by  $\nu|_{\mathfrak{a}} = -2a \text{id}_{\mathfrak{a}}$ ,  $\nu(e_i) = 0$  for  $i = 3, 4, 5, 6$ , we may shear  $\varphi$  to a calibrated  $G_2$ -structure on

$$(-a.17 - 2a.(36 + 45), -a.27 - 2a.(35 - 46), b.37, b.47, c.57, c.67, 0).$$

Case (a)(ii): We assume now that  $a = 0$  and take  $\mathfrak{a} = \text{span}(e_1, e_2)$ ,  $a_1, a_2, a_3 \in \mathbb{R}$  with  $a_1^2 + a_2a_3 = 0$ ,

$$\omega_0 = (a_1(e^{35} - e^{46}) + a_2(e^{36} + e^{45})) \otimes e_1 - (a_3(e^{35} - e^{46}) - a_1(e^{36} + e^{45})) \otimes e_2$$

and  $\nu \in \mathfrak{u}^* \otimes \mathfrak{a}$  defined by  $\nu(e_1) = a_3e_1 + a_1e_2$ ,  $\nu(e_2) = a_1e_1 - a_2e_2$  and  $\nu(e_i) = 0$  for all  $i = 3, \dots, 6$ . With this data, we may shear  $\varphi$  to a calibrated  $G_2$ -structure on

$$(a_3.17 - a_1.(35 - 46 - 27) - a_2.(36 + 45), \\ -a_2.27 + a_3.(35 - 46) - a_1.(36 + 45 - 17), b.37, b.47, -b.57, -b.67, 0).$$

Case (b): Taking  $\mathfrak{a} = \text{span}(e_1, e_2, e_3, e_4)$ ,  $\omega_0 = -e^{56} \otimes e_1$  and  $\nu \in \text{End}(\mathfrak{u})$  given by  $\nu|_{\text{span}(e_1, e_2)} = -(3a + 2b) \text{id}_{\text{span}(e_1, e_2)}$ ,  $\nu|_{\text{span}(e_3, e_4)} = (3a + 2b) \text{id}_{\text{span}(e_3, e_4)}$ ,  $\nu(e_5) = \frac{1}{2}e_3$  and  $\nu(e_6) = -\frac{1}{2}e_4$ , we may shear  $\varphi$  to a calibrated  $G_2$ -structure on

$$(2c.17 + 56, 2c.27, -3c.37 + \frac{1}{2}.57, -3c.47 - \frac{1}{2}.67, c.57, c.67, 0).$$

Case (c): Taking  $\mathfrak{a} = \text{span}(e_1, e_2, e_4, e_5)$ ,  $\omega_0 = 4ae^{36} \otimes e_1$  and  $\nu \in \text{End}(\mathfrak{u})$  with  $\nu(e_1) = -2ae_1$ ,  $\nu(e_2) = -4ae_2$ ,  $\nu|_{\text{span}(e_4, e_5)} = 2a \text{id}_{\text{span}(e_4, e_5)}$  and  $\nu|_{\text{span}(e_3, e_6)} = 0$ , we may shear  $\varphi$  to a calibrated  $G_2$ -structure on

$$(-a.17 - 4a.36, -3a.27, b.37, (b + 2a).47, (a - b).57, -(a + b).67, 0).$$

Note that in cases (b) and (c) of Proposition 4.6, the shear Lie algebra is of the form  $(\mathfrak{h}_3 \oplus \mathbb{R}^3) \rtimes \mathbb{R}$ , where  $\mathfrak{h}_3$  is the three-dimensional Heisenberg algebra. We thus obtain calibrated  $G_2$ -structures on such Lie algebras as the shears of calibrated  $G_2$ -structures on almost Abelian Lie algebras. In fact, we get all possible calibrated  $G_2$ -structures on that class of Lie algebras this way:

**Corollary 4.8.** *Let  $\mathfrak{g}$  be a seven-dimensional Lie algebra with a codimension one nilpotent ideal  $\mathfrak{u} \cong \mathfrak{h}_3 \oplus \mathbb{R}^3$ . Let  $\varphi \in \Lambda^3 \mathfrak{g}^*$  be a  $G_2$ -structure. Fix  $X \perp \mathfrak{u}$  with  $\|X\| = 1$ .*

*Write  $h = \text{ad}(X)|_{\mathfrak{u}} \in \mathfrak{der}(\mathfrak{u})$  and  $(g, J, \sigma, \rho)$  for the special almost Hermitian structure on  $\mathfrak{u}$  induced by  $\varphi$ . Set*

$$U_1 = [\mathfrak{u}, \mathfrak{u}] \oplus J[\mathfrak{u}, \mathfrak{u}], \quad U_2 = U_1^\perp \cap \mathfrak{z}(\mathfrak{u}).$$

*Then  $\varphi \in \Lambda^3 \mathfrak{g}^*$  is calibrated if and only if  $J[\mathfrak{u}, \mathfrak{u}] \subset \mathfrak{z}(\mathfrak{u})$  and either*

- (i)  $\mathfrak{z}(\mathfrak{u})$  is  $J$ -invariant and there are  $\lambda, \mu \in \mathbb{R}$  and linear maps  $h_{ij}: U_j \rightarrow U_i$  with  $[h_{ij}, J] = 0$  such that

$$h = \begin{pmatrix} -2\lambda & h_{12} & h_{13} \\ 0 & 3\lambda + \mu J & h_{23} + h \\ 0 & 0 & -\lambda - \mu J \end{pmatrix}$$

*on  $\mathfrak{u} = U_1 \oplus U_2 \oplus U_3$ , where  $U_3 := \mathfrak{z}(\mathfrak{u})^\perp$  and  $h: U_3 \rightarrow U_2$  is given by*

$$h(Z) = -\frac{\|[Z, JZ]\|^2}{2\|\rho([Z, JZ], Z, \cdot)\|^2} \rho([Z, JZ], Z, \cdot)^\sharp,$$

*for all  $Z \in U_3$ , or*

- (ii)  $\mathfrak{z}(\mathfrak{u})$  is not  $J$ -invariant,  $\rho$  is zero on  $[\mathfrak{u}, \mathfrak{u}] \wedge \Lambda^2 \mathfrak{z}(\mathfrak{u})$  and there is an  $h_1 \in \mathfrak{sl}(\mathfrak{u}, \rho)$  with  $U_1 \subset \ker h_1$  and  $h_1(\mathfrak{z}(\mathfrak{u})) \subset \mathfrak{z}(\mathfrak{u})$  such that

$$h = \text{diag}(-2\lambda, -6\lambda, 3\lambda, -\lambda) + h_1$$

on  $\mathfrak{u} = [\mathfrak{u}, \mathfrak{u}] \oplus J[\mathfrak{u}, \mathfrak{u}] \oplus U_2 \oplus JU_2$  with  $\lambda \in \mathbb{R}$  specified by  $-8\lambda\rho(Z_1, Z_2, \cdot)^\sharp = J[JZ_1, JZ_2]$  for any basis  $Z_1, Z_2$  of  $U_2$ .

*Proof.* The derivations  $h$  of  $\mathfrak{u}$  on the shear obtained from Proposition 4.6 (b) and (c) are exactly those given in Corollary 4.8 (i) and (ii): this may be seen by straightforward computations using  $h = f + \tilde{\nu} + \hat{\nu}$  and  $[W_1, W_2] = \tilde{\omega}(W_1, W_2)Y$ , for any  $W_1, W_2 \in \mathfrak{z}(\mathfrak{u})^\perp$ , and for (ii),  $\rho(Z_1, Z_2, \cdot)^\sharp \in \text{span}(JY)$  in Proposition 4.6(c). So the direction “ $\Leftarrow$ ” follows.

For the converse direction, we show that we can shear, with left-invariant data  $(\text{inc}, \text{id}_{\mathfrak{a}}, \omega_0)$ , any calibrated  $G_2$ -structure  $\varphi$  on an almost nilpotent Lie algebra of the form  $(\mathfrak{h}_3 \oplus \mathbb{R}^3) \rtimes \mathbb{R}$  to one on an almost Abelian Lie algebra. As a left-invariant shear can be inverted by Theorem 3.14, we may obtain  $\varphi$  as the shear of a calibrated  $G_2$ -structure on an almost Abelian Lie algebra. Now Proposition 4.6 (b) and (c) contain all possible calibrated shears of calibrated almost Abelian Lie algebras to Lie algebras of the form  $(\mathfrak{h}_3 \oplus \mathbb{R}^3) \rtimes \mathbb{R}$  provided we have  $\mathfrak{a} = \ker(\omega_0)$ . However, if  $\mathfrak{a} \subset \ker(\omega_0)$ , we may simply enlarge  $\mathfrak{a}$  to the  $f$ -invariant subspace  $\ker(\omega_0)$  and the direction “ $\Rightarrow$ ” then follows.

So let  $\varphi$  be calibrated and note that  $\mathfrak{z}(\mathfrak{u})$  is an  $h$ -invariant subspace of  $\mathfrak{u}$ . For the shear, we take  $\mathfrak{a} = \mathfrak{z}(\mathfrak{u})$  and  $\omega = \omega_0 + \alpha \wedge \nu$  with  $\omega_0 \in \Lambda^2 \mathfrak{u}^* \otimes \mathfrak{a}$ ,  $\nu \in \mathfrak{u}^* \otimes \mathfrak{a} \subset \text{End}(\mathfrak{u})$  and  $\alpha \in \mathfrak{g}^*$  uniquely defined by  $\alpha(X) = 1$  and  $\alpha(\mathfrak{u}) = \{0\}$ . To shear to an almost Abelian Lie algebra, we must take  $\omega_0 = -[\text{proj}_{\mathfrak{u}}(\cdot), \text{proj}_{\mathfrak{u}}(\cdot)]$ . Then  $\text{im}(\omega_0) = [\mathfrak{u}, \mathfrak{u}]$ . Similarly to the proof of Proposition 4.2 and using that  $\mathfrak{a}$  is central, we get  $\gamma = \alpha \otimes h|_{\mathfrak{a}}$  and  $\eta = \gamma - \omega|_{\mathfrak{a} \otimes \mathfrak{g}} = \alpha \otimes (h + \nu)|_{\mathfrak{a}}$ . Furthermore, the Jacobi identity gives us  $d\omega_0|_{\Lambda^3 \mathfrak{u}} = 0$ , so  $d\omega_0 = \alpha \wedge h.\omega_0$ . Hence,  $(\text{inc}, \text{id}_{\mathfrak{a}}, \omega)$  defines left-invariant shear data on  $G$  if and only if

$$\begin{aligned} 0 &= d\omega + \eta \wedge \omega = \alpha \wedge h.\omega_0 - \alpha \wedge d_{\mathfrak{u}}\nu + \alpha \wedge (h + \nu) \circ \omega_0 \\ &= \alpha \wedge (h.\omega_0 - d_{\mathfrak{u}}\nu + (h + \nu) \circ \omega_0), \end{aligned}$$

giving  $h.\omega_0 - d_{\mathfrak{u}}\nu + (h + \nu) \circ \omega_0 = 0$ , where  $d_{\mathfrak{u}}$  is the differential of  $\mathfrak{u}$ . However,  $h$  is a derivation, so

$$(h.\omega_0 - d_{\mathfrak{u}}\nu)(Z_1, Z_2) = [h(Z_1), Z_2] + [Z_1, h(Z_2)] + \nu[Z_1, Z_2] = -((h + \nu) \circ \omega_0)(Z_1, Z_2)$$

for all  $Z_1, Z_2 \in \mathfrak{u}$ . Thus  $(\text{inc}, \text{id}_{\mathfrak{a}}, \omega)$  always defines left-invariant shear data on  $G$ .

(i) Here, we set  $\nu|_{\mathfrak{a}} = 0$  and  $\nu(Z) = \frac{\|[Z, JZ]\|^2}{2\|\rho([Z, JZ], Z, \cdot)\|^2} \rho([Z, JZ], Z, \cdot)^\sharp$  for any  $Z \in \mathfrak{a}^\perp \subset \mathfrak{u}$ . The shear is again calibrated if and only if (4.3) holds. The first equation in (4.3) is equivalent to the vanishing of the anti-symmetrisation of  $\rho([\cdot, \cdot], \cdot, \cdot)$ , and so is satisfied since  $[\mathfrak{u}, \mathfrak{u}] \subset \mathfrak{z}(\mathfrak{u})$  and  $\mathfrak{z}(\mathfrak{u})$  is four-dimensional and  $J$ -invariant. The second equation in (4.3) is given by  $\gamma = \nu.\rho$  with  $\gamma(X, Y, Z) = \sum_{\text{cyclic}} g([X, Y], JZ)$  for  $X, Y, Z \in \mathfrak{u}$ . As  $\nu(\mathfrak{u}) \subset \mathfrak{z}(\mathfrak{u}) \cap ([\mathfrak{u}, \mathfrak{u}] \oplus J[\mathfrak{u}, \mathfrak{u}])^\perp$ , both sides of the equation are zero on  $\Lambda^2 \mathfrak{z}(\mathfrak{u}) \wedge \mathfrak{u} + \Lambda^3([\mathfrak{u}, \mathfrak{u}] \oplus J[\mathfrak{u}, \mathfrak{u}])^\perp$ . Finally, a straightforward computation yields

$$(\nu.\rho)(Y, Z, JZ) = \frac{\|[Z, JZ]\|^2}{\|\rho([Z, JZ], Z, \cdot)\|^2} g(\rho(JY, Z, \cdot), \rho([Z, JZ], Z, \cdot))$$

for any  $Y \in [\mathfrak{u}, \mathfrak{u}] \oplus J[\mathfrak{u}, \mathfrak{u}]$  and any  $Z \in \mathfrak{z}(\mathfrak{u})^\perp$ . For  $Y \in [\mathfrak{u}, \mathfrak{u}]$ , the right-hand side is zero and for  $Y \in J[\mathfrak{u}, \mathfrak{u}]$ , one has  $JY \in \text{span}([Z, JZ])$  and so  $\nu.\rho(Y, Z, JZ) = g([Z, JZ], JY) = \gamma(Y, Z, JZ)$  as we wanted. Hence, the shear is calibrated.

(ii) Inserting into  $0 = d\varphi$  two non-zero elements of  $\mathfrak{z}(\mathfrak{u})^\perp \subset \mathfrak{u}$  and two elements of  $\mathfrak{z}(\mathfrak{u})$ , we obtain  $\rho|_{[\mathfrak{u}, \mathfrak{u}] \wedge \Lambda^2 \mathfrak{z}(\mathfrak{u})} = 0$ . This implies  $J[\mathfrak{u}, \mathfrak{u}] \subset \mathfrak{z}(\mathfrak{u})$  as otherwise we

may take  $Y \in [\mathfrak{u}, \mathfrak{u}]$ ,  $Z_1 \in \mathfrak{z}(\mathfrak{u}) \cap J\mathfrak{z}(\mathfrak{u})$  and  $Z_2 \in \mathfrak{z}(\mathfrak{u})$  such that  $Y, Z_1, Z_2$  is a  $\mathbb{C}$ -basis of  $\mathfrak{u}$  and so must have  $0 \neq \rho(Y, Z_1, Z_2)$  or  $0 \neq \rho(Y, JZ_1, Z_2)$ , a contradiction. After these preliminary considerations, set  $U := \mathfrak{z}(\mathfrak{u}) \cap ([\mathfrak{u}, \mathfrak{u}] \oplus J[\mathfrak{u}, \mathfrak{u}])^\perp$ , define  $\lambda \in \mathbb{R}$  via the formula  $4\lambda\rho(Z_1, Z_2, \cdot)^\sharp = -J[Z_1, Z_2]$ , where  $Z_1, Z_2$  is any basis of  $JU$ , and define  $\nu \in \mathfrak{u}^* \otimes \mathfrak{a}$  by  $\nu(Y) = -2\lambda Y$ ,  $\nu(JY) = -4\lambda JY$  for all  $Y \in [\mathfrak{u}, \mathfrak{u}]$ ,  $\nu(Z) = 2\lambda Z$  for all  $Z \in U$  and  $\nu|_{JU} = 0$ . Firstly,  $\rho|_{[\mathfrak{u}, \mathfrak{u}] \wedge \Lambda^2 \mathfrak{z}(\mathfrak{u})} = 0$  implies that the anti-symmetrisation of  $\rho([\cdot, \cdot], \cdot, \cdot)$  vanishes. Furthermore, both  $\nu \cdot \rho$  and  $\gamma$  as above, are zero on  $\Lambda^3 \mathfrak{z}(\mathfrak{u}) + (JY^\perp \cap \mathfrak{z}(\mathfrak{u})) \wedge \Lambda^2 \mathfrak{u}$ . Finally, for  $Y \in [\mathfrak{u}, \mathfrak{u}]$  and  $Z_1, Z_2 \in JU$  we get

$$\begin{aligned} (\nu \cdot \rho)(JY, Z_1, Z_2) &= 4\lambda\rho(JY, Z_1, Z_2) = g(4\lambda\rho(Z_1, Z_2, \cdot)^\sharp, JY) \\ &= -g(J[Z_1, Z_2], JY) = g([Z_1, Z_2], J(JY)) = \gamma(JY, Z_1, Z_2), \end{aligned}$$

as required.  $\square$

**4.3.3. Almost semi-Kähler structures.** An almost Hermitian structure  $(g, J, \sigma)$  on a  $2n$ -dimensional manifold is called *almost semi-Kähler* if  $d(\sigma^{n-1}) = 0$ . Suppose  $(g, J, \sigma)$  is an almost semi-Kähler structure on a  $2n$ -dimensional almost Abelian Lie algebra  $(\mathfrak{g}, \mathfrak{u})$ . Fix a unit vector  $X \in \mathfrak{u}^\perp \subset \mathfrak{g}$  and let  $\alpha \in \mathfrak{u}^\circ \subset \mathfrak{g}^*$  be the element with  $\alpha(X) = 1$ . Then  $\sigma = (JX)^b \wedge \alpha + \sigma_1$  for some  $\sigma_1$  with kernel  $\text{span}(X, JX)$ . Thus

$$\sigma^{n-1} = (n-1)(JX)^b \wedge \sigma_1^{n-2} \wedge \alpha + \sigma_1^{n-1}$$

and the almost semi-Kähler condition is equivalent to  $f \cdot \sigma_1^{n-1} = 0$  for  $f = \text{ad}(X)|_{\mathfrak{u}}$ . Since  $\sigma_1^{n-1}$  defines a volume form on  $U := \text{span}(X, JX)^\perp$ , this is the same as  $f(JX) = \text{tr}(f)JX$ .

The first equation in (4.3) is always satisfied, as  $\sigma_1^{n-1} \wedge \omega_0$  is an  $n$ -form with values in  $\mathfrak{a}$  on the  $(n-1)$ -dimensional vector space  $\mathfrak{u}$ .

Let us consider the case  $\text{im}(\omega_0) = \text{span}(JX) \subset \ker(\omega_0) = \mathfrak{a}$  with  $\mathfrak{a}$  an  $f$ -invariant subspace of  $\mathfrak{u}$ . Then  $\omega_0 = \tilde{\omega} \otimes JX$  for a non-zero  $\tilde{\omega} \in \Lambda^2 \mathfrak{u}^*$  and the first equation in (4.2) is equivalent to  $f \cdot \tilde{\omega} = \lambda \tilde{\omega}$  and  $\nu(JX) = -(\lambda + \text{tr}(f))JX$ , for some  $\lambda \in \mathbb{R}$ . The second equation in (4.3) reads  $\tilde{\omega} \wedge \sigma_1^{n-2} = \nu \cdot \sigma_1 \wedge \sigma_1^{n-2}$ . This is fulfilled if and only if  $\tilde{\omega} - \nu \cdot \sigma_1 \in [\Lambda_0^{1,1} U^*] \oplus [\Lambda^{2,0} U^*]$ . Hence, if, e.g.,  $\tilde{\omega} \in [\Lambda_0^{1,1} U^*] \oplus [\Lambda^{2,0} U^*]$  is such that  $f \cdot \tilde{\omega} = \lambda \tilde{\omega}$  for some  $\lambda \in \mathbb{R}$ , then one obtains left-invariant shear data with the shear being again almost semi-Kähler by taking  $\nu \in \text{End}(\mathfrak{u})$  with  $\nu|_U = 0$  and  $\nu(JX) = -(\lambda + \text{tr}(f))JX$ .

**Example 4.9.** To get an explicit example, take the Lie algebra

$$(a_1.16, a_2.26, a_3.36, a_4.46, a_5.56, 0), \quad \sum_{i=1}^4 a_i = 0.$$

Then  $\mathfrak{g}$  admits an almost semi-Kähler structure with  $\sigma = 12 + 34 + 56$ . One choice of shear is via  $\omega_0 = -e^{13} \otimes e_5$ ,  $\nu = (a_1 + a_3 - a_5)e^5 \otimes e_5$ , giving an almost semi-Kähler structure on

$$(a_1.16, a_2.26, a_3.36, a_4.46, (a_1 + a_3).56 + 13, 0).$$

For other choices, take  $\tilde{\omega}$  to be  $e^{14}$ ,  $e^{23}$  or  $e^{24}$ .

Another example may be obtained if  $a_1 = a_2 = -a_3$ . Then  $(\mathfrak{g}, \sigma)$  is even *semi-Kähler*, i.e.  $J$  is integrable. Moreover, if we shear  $(\mathfrak{g}, \sigma)$  with  $\omega_0 = -(e^{13} + e^{24}) \otimes e_5$ ,  $\nu = -a_5 e^5 \otimes e_5$ , we get a semi-Kähler structure on

$$(a_1.16, a_1.26, -a_1.36, -a_1.46, 13 + 24, 0)$$

by the above and Proposition 3.13 as  $e^{13} + e^{24}$  is a  $(1, 1)$ -form on  $(\mathfrak{g}, J)$ .

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